An aymptotic minimal contractor for non-linear equations provided in the Codac library

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Section 1

Introduction

Introduction

Problem statement

We consider the problem of approximating the solutions of the system:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^p$ is a non-linear and differentiable function. Possibly non continuous.

In particular, we will look at systems where:

$$p < n$$

..for which the solution set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$ has infinitely many solutions.

Introduction

Set-inversion problem

Set inversion can be done with

- **forward** evaluations of $\mathbf{f}$
  (see *e.g.* SIVIA)
  - limits in high dimensions due to bisections

- **forward**/reverse evaluations of $\mathbf{f}$
  (such as contractors + pavers)
  - lower complexity, bisections used as a last resort
  - efficient for constraint propagation

Introduction

Fwd/bwd evaluations

Forward evaluations:

`fwd_plus`($[x]$,$[y]$): returns $[\{z \mid \exists x \in [x], \exists y \in [y], z = x + y\}]$

Nathalie Revol. Introduction to the IEEE 1788-2015 standard for interval arithmetic. International Workshop on Numerical Software Verification. pp. 14-21, Springer, Germany (2017)

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Reverse (backward) evaluations:

`bwd_plus`([z],[x],[y]): returns $[\{(x, y) \in ([x], [y]) \mid x + y \in [z]\}]$

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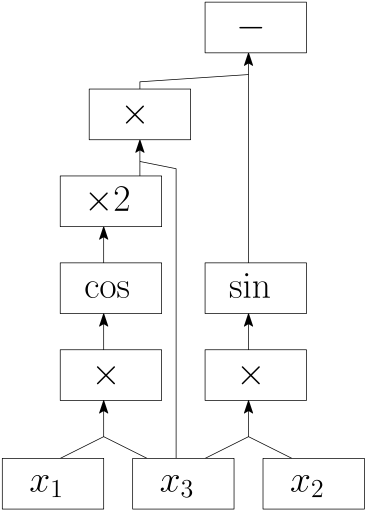
Section 2

The HC4Revise algorithm

The HC4Revise algorithm

A "forward-backward" algorithm

DAG associated with:
$f(\mathbf{x}) = 2x_3 \cos(x_1 x_3) - \sin(x_2 x_3)$



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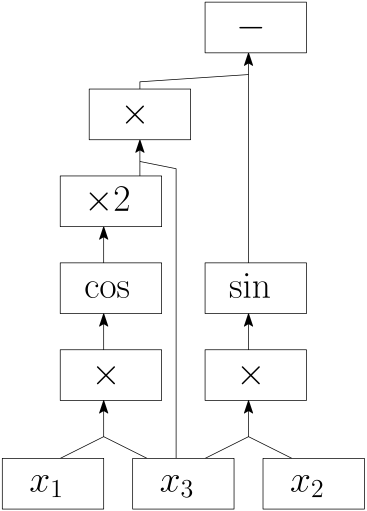

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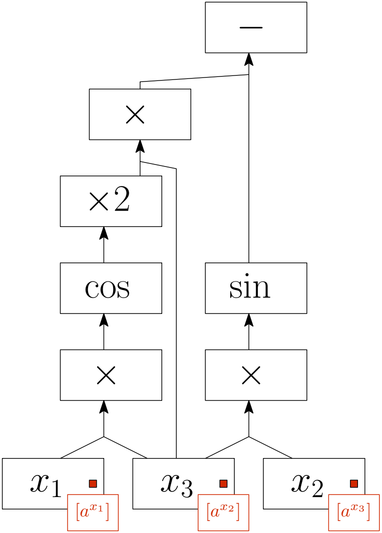

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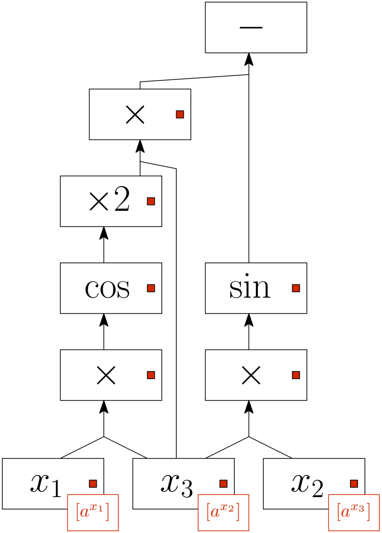

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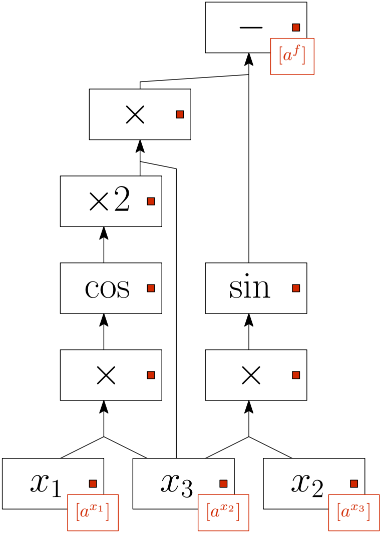

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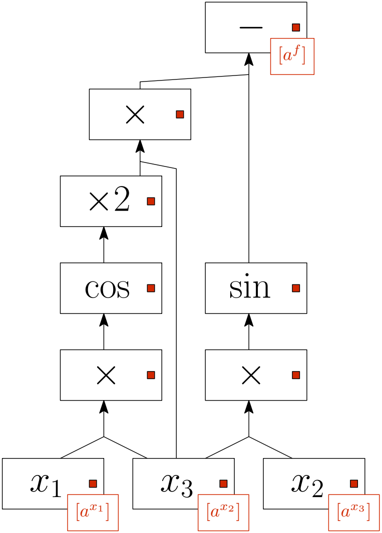

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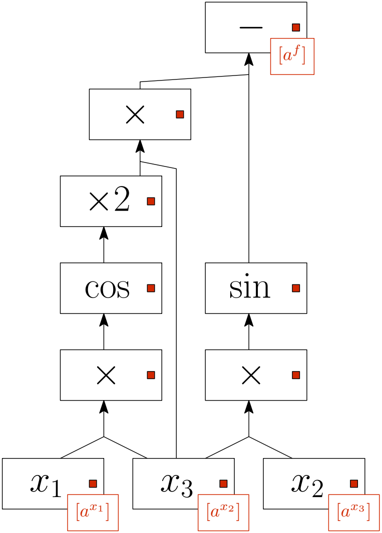
5. possible contraction of $([a^{x_1}], [a^{x_2}], [a^{x_3}])$



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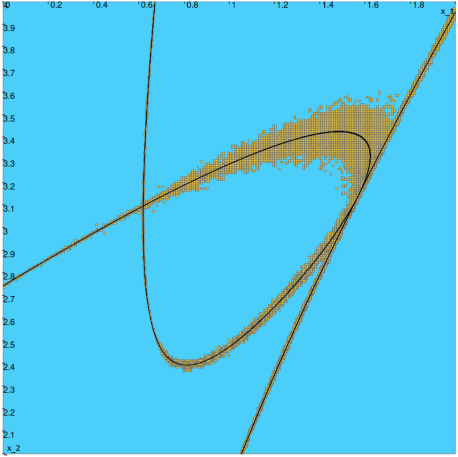
Limits of the algorithm

$$\begin{pmatrix} -x_3^2 + 2x_3\sin(x_3x_1) + \cos(x_3x_2) \\ 2x_3\cos(x_3x_1) - \sin(x_3x_2) \end{pmatrix} = \mathbf{0}$$



Example from:

R. Malti, M. Rapaić, and V. Turkulov. A unified framework for robust stability analysis of linear irrational systems in the parametric space. Annual Reviews in Control, vol. 57, 2024.

- delayed systems
- $\mathbf{x} \in [0,2]\times[2,4]\times[0,10]$
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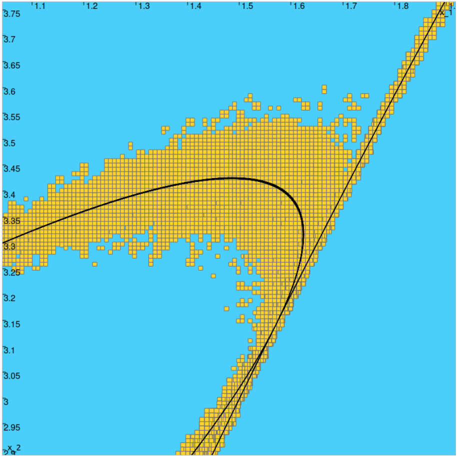

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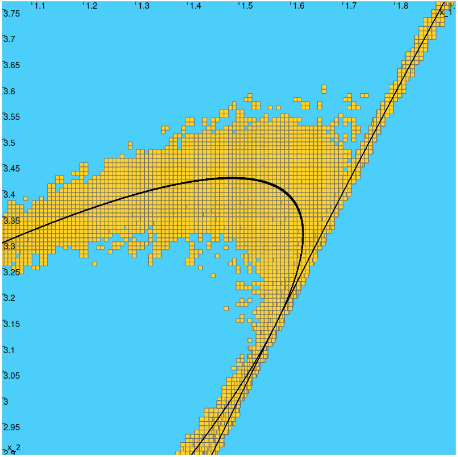

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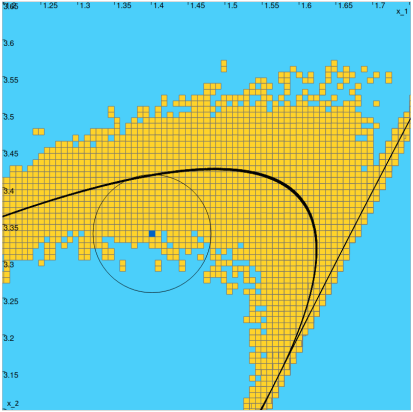
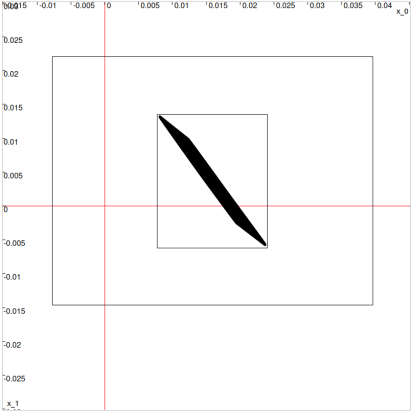
Two problems

The HC4Revise algorithm

Clustering effect due to: [1] dependency problem



$\mathbb{R}^n$ space (projection)

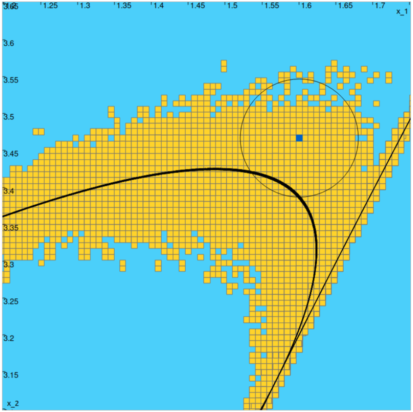


$\mathbb{R}^p$ space

$\mathbf{f}$

The HC4Revise algorithm

Clustering effect due to: [2] wrapping effect



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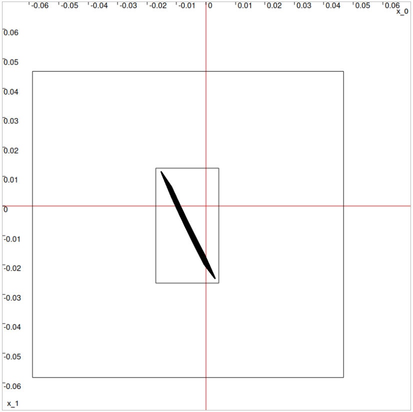

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The HC4Revise algorithm

Solutions for these clustering effects

For [1]: dependency problem

For [2]: wrapping effect

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- centered form provides minimal results with narrow boxes

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These solutions come from a recent contribution with proof:

`L. Jaulin (2024). Asymptotically minimal interval contractors based on the centered form; Application to the stability analysis of linear time-delayed differential equations, Acta Cybernetica.`

Section 3

A contractor involving the centered form

A contractor involving the centered form

Forward evaluations with the "centered form"

The classical formula is given as:

$$[\mathbf{f}_c]([\mathbf{x}]) = \mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J_f}]\left([\mathbf{x}]\right) \cdot ([\mathbf{x}] - \overline{\mathbf{x}})$$

with $\overline{\mathbf{x}}$ the center of the box $[\mathbf{x}]$,
and $[\mathbf{J_f}]\left([\mathbf{x}]\right)$ the interval Jacobian matrix of $\mathbf{f}$ evaluated over $[\mathbf{x}]$.

R. Moore, Methods and Applications of Interval Analysis
Society for Industrial and Applied Mathematics, jan 1979.

- traditionally used to enclose the range of a function over narrow intervals
- asymptotically small overestimation for sufficiently narrow boxes on scalar functions

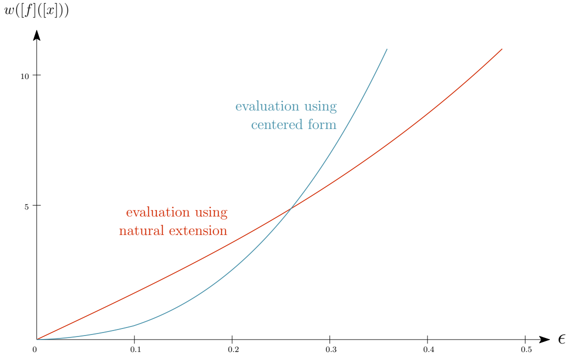
A contractor involving the centered form

The dependency problem: example

Consider for instance this expression with multiple occurrences of x:

$$f(x) = 2x^5 + x^3 - 3x^2$$

Let us try the evaluation for several growing inputs:
$[x] = 0.7 + [-\epsilon, \epsilon]$, $\epsilon = 0 \dots 0.5$



Comparing pessimism: natural evaluation (red) *vs* centered form evaluation (blue).

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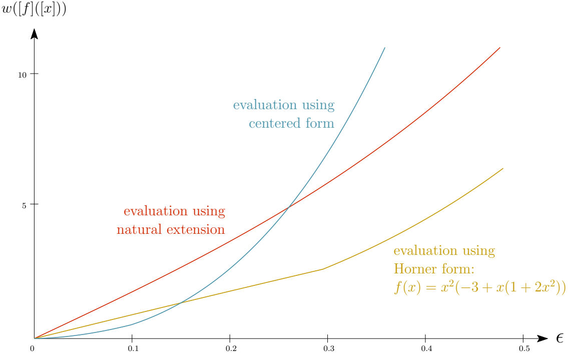

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The constraint $\{\mathbf{f}(\mathbf{x}) \in [\mathbf{y}],\ \mathbf{x} \in [\mathbf{x}]\}$ is then expressed using a centered form expression:

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terms to be computed simultaneously during the forward DAG evaluation

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Simultaneous evaluation $\Rightarrow$ operator overloading

A contractor involving the centered form

Interval Automatic Differentiation (IAD)

Contribution of this work:

We use IAD to automatically compute the term $[\mathbf{J_f}]([\mathbf{x}])$.

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About Automatic Differentiation:

- techniques to automatically evaluate the partial derivatives of a function
- `HC4Revise` $\Rightarrow$ execution of a sequence of elementary algorithms:

  `fwd_plus`, `fwd_cos`, *etc* (and their reverse counterparts)
- using **chain rule** and interval analysis: compute AD for approximating $[\mathbf{J_f}]([\mathbf{x}])$

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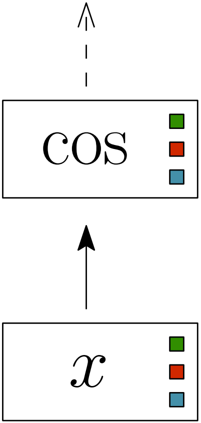
$\Rightarrow$ operator overloading in programming languages

A contractor involving the centered form

Examples of operator overloading



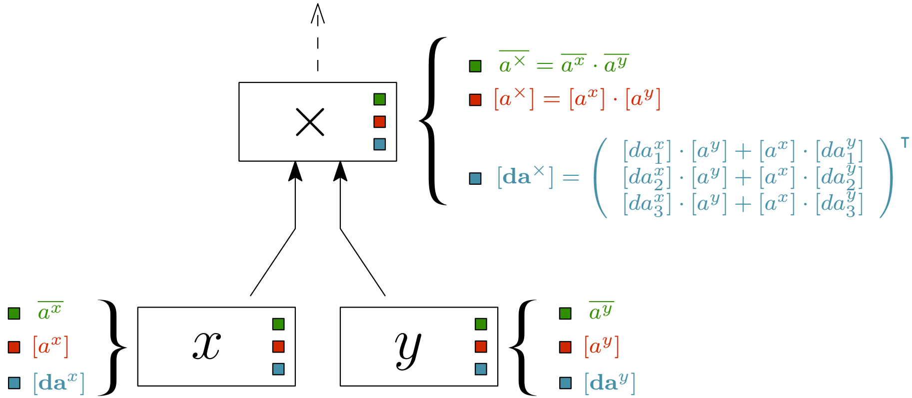
- $\overline{a^{\cos}} = \cos(\overline{a^x})$
- $[a^{\cos}] = \cos([a^x])$
- $[\mathbf{da}^{\cos}] = \begin{pmatrix} -\sin([a^x]) \cdot [da_1^x] \\ -\sin([a^x]) \cdot [da_2^x] \\ -\sin([a^x]) \cdot [da_3^x] \end{pmatrix}^\intercal$

- $\overline{a^x}$
- $[a^x]$
- $[\mathbf{da}^x]$

Overloading the cos operator with automatic differentiation

A contractor involving the centered form

Examples of operator overloading



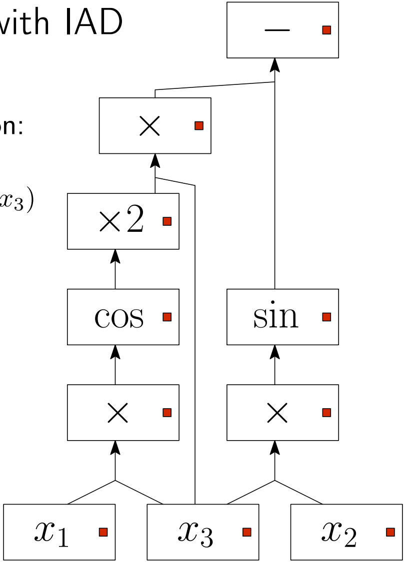
Overloading the product operator with automatic differentiation

A contractor involving the centered form

Directed acyclic graph, now with IAD

Directed acyclic graph (DAG)
involving automatic differentiation:

Ex: $f(\mathbf{x}) = 2x_3\cos(x_1x_3) - \sin(x_2x_3)$



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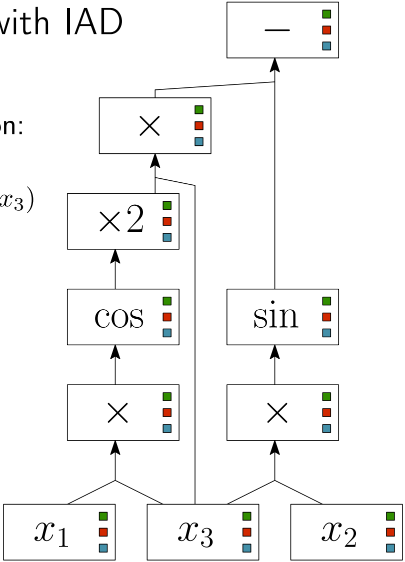

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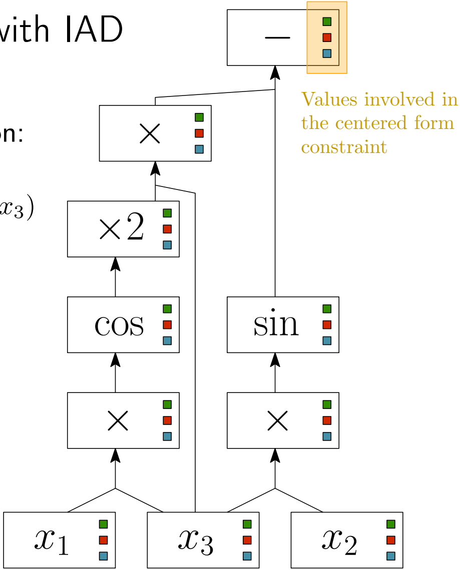

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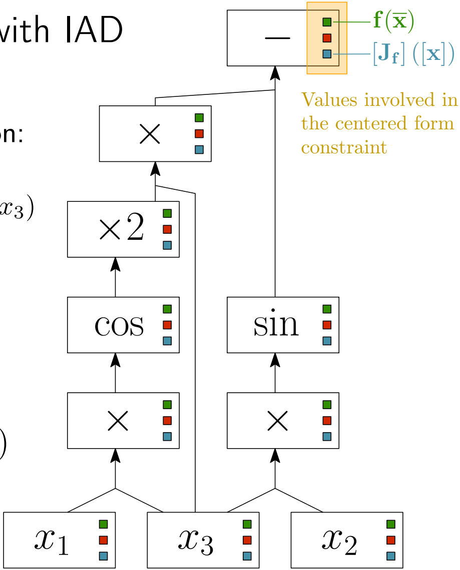
Towards centered form for forward interval evaluation of f

$$\mathbf{f}([\mathbf{x}]) \subset \mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J_f}]([\mathbf{x}]) \cdot ([\mathbf{x}] - \overline{\mathbf{x}})$$



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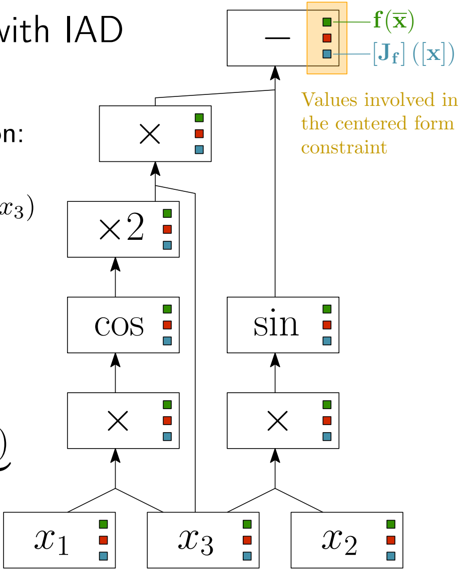
Directed acyclic graph, now with IAD

Directed acyclic graph (DAG) involving automatic differentiation:

Ex: $f(\mathbf{x}) = 2x_3\cos(x_1x_3) - \sin(x_2x_3)$

Towards centered form for forward interval evaluation of f

$$\mathbf{f}([\mathbf{x}]) \subset \underbrace{\mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J_f}]([\mathbf{x}]) \cdot ([\mathbf{x}] - \overline{\mathbf{x}})}_{\text{linear system}}$$



Section 4

Top level algorithm for the `CtcInverse` contractor

Top level algorithm for the `CtcInverse` contractor

Recall the constraint $\{\mathbf{f}(\mathbf{x}) \in [\mathbf{y}],\ \mathbf{x} \in [\mathbf{x}]\}$, expressed using a centered form expression:

$$\mathbf{f}(\mathbf{x}) \in [\mathbf{y}] \cap \left(\mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J}_{\mathbf{f}}]\big([\mathbf{x}]\big)\big([\mathbf{x}] - \overline{\mathbf{x}}\big)\right)$$

Top level algorithm for the `CtcInverse` contractor

Recall the constraint $\{\mathbf{f}(\mathbf{x}) \in [\mathbf{y}],\ \mathbf{x} \in [\mathbf{x}]\}$, expressed using a centered form expression:

$$\mathbf{f}(\mathbf{x}) \in [\mathbf{y}] \cap \Big(\mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J}_{\mathbf{f}}]\big([\mathbf{x}]\big)\big([\mathbf{x}] - \overline{\mathbf{x}}\big)\Big) \cap [\mathbf{f}]([\mathbf{x}])$$

Top level algorithm for the `CtcInverse` contractor

Recall the constraint $\{\mathbf{f}(\mathbf{x}) \in [\mathbf{y}],\ \mathbf{x} \in [\mathbf{x}]\}$, expressed using a centered form expression:

$$\mathbf{f}(\mathbf{x}) \in [\mathbf{y}] \cap \left(\mathbf{f}(\overline{\mathbf{x}}) + [\mathbf{J}_\mathbf{f}]\big([\mathbf{x}]\big)\big([\mathbf{x}] - \overline{\mathbf{x}}\big)\right) \cap [\mathbf{f}]([\mathbf{x}])$$

This can be expressed under the form of a linear constraint:

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{p}$$

with

- $\mathbf{b} \in ([\mathbf{y}] \cap [\mathbf{f}]([\mathbf{x}])) - \mathbf{f}(\overline{\mathbf{x}}),\ \in \mathbb{IR}^p$
- $\mathbf{A} \in [\mathbf{J}_\mathbf{f}]\big([\mathbf{x}]\big),\ \in \mathbb{IR}^{p \times n}$
- $\mathbf{p} \in [\mathbf{x}] - \overline{\mathbf{x}},\ \in \mathbb{IR}^n$

Top level algorithm for the `CtcInverse` contractor

Resulting algorithm for the contractor involving the centered form, for dealing with the constraint $\mathbf{f}(\mathbf{x}) \in [\mathbf{y}]$

$\texttt{CtcInverse}(in: \ \mathbf{f}, [\mathbf{y}], \ in/out: \ [\mathbf{x}])$

$$
\begin{aligned}
&\overline{\mathbf{x}} \leftarrow \mathrm{mid}([\mathbf{x}]) \\
&\left(\mathbf{a}^{\mathbf{f}}, [\mathbf{a}^{\mathbf{f}}], [\mathbf{da}^{\mathbf{f}}]\right) \leftarrow \texttt{fwd\_dag\_eval}(\mathbf{f}, [\mathbf{x}], \overline{\mathbf{x}}) \\
&[\mathbf{p}] \leftarrow [\mathbf{x}] - \overline{\mathbf{x}} \\
&[\mathbf{b}] \leftarrow \left([\mathbf{y}] \cap [\mathbf{a}^{\mathbf{f}}]\right) - \mathbf{a}^{\mathbf{f}} \\
&[\mathbf{A}] \leftarrow [\mathbf{da}^{\mathbf{f}}] \\
&([\mathbf{A}], [\mathbf{p}]) \leftarrow \texttt{reverse\_mul}([\mathbf{b}], [\mathbf{A}], [\mathbf{p}]) \\
&[\mathbf{x}] \leftarrow [\mathbf{p}] + \overline{\mathbf{x}}
\end{aligned}
$$

with the terms $\mathbf{a}^{\mathbf{f}}$, $[\mathbf{a}^{\mathbf{f}}]$ and $[\mathbf{da}^{\mathbf{f}}]$ computed during the forward evaluation of the DAG.

Top level algorithm for the `CtcInverse` contractor

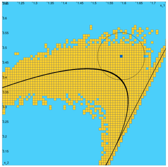
Dealing efficiently with the linear constraint

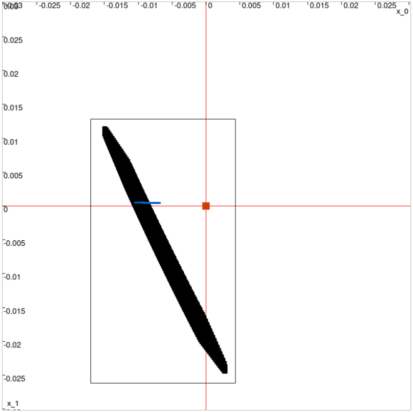
About the **reverse matrix-vector product**:

$([\mathbf{A}], [\mathbf{p}]) \leftarrow \texttt{reverse\_mul}([\mathbf{b}], [\mathbf{A}], [\mathbf{p}])$

Some preconditioning allows to significantly reduce the wrapping effect.

This can be achieved using a Gauss Jordan band diagonalization method.



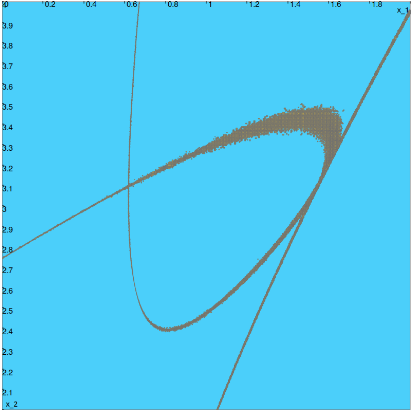


Section 5

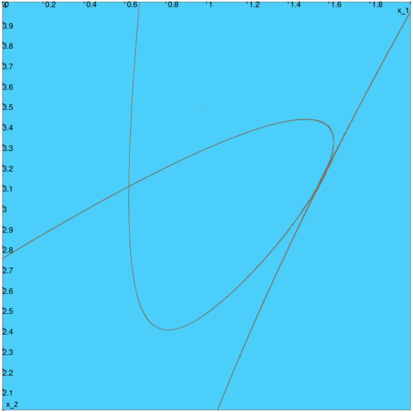
Results of `CtcInverse` provided in the Codac library

Results of `CtcInverse` provided in the Codac library

Previous example using the centered form



$\mathbb{X}$ computed with `HC4Revise`.
Computation time: 4.51s. 27430 boxes.

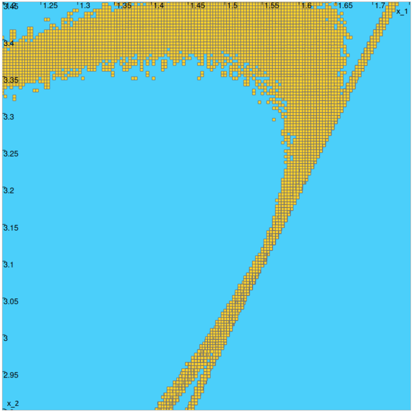


$\mathbb{X}$ computed with `CtcInverse`.
Computation time: 0.69s. 3713 boxes.

Projection of the boxes $[\mathbf{x}] \in \mathbb{IR}^3$ onto (x_1, x_2).

Results of `CtcInverse` provided in the Codac library

Previous example using the centered form



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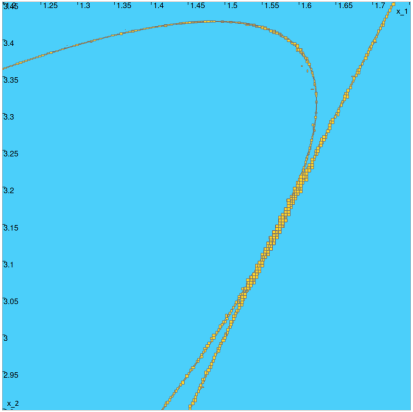

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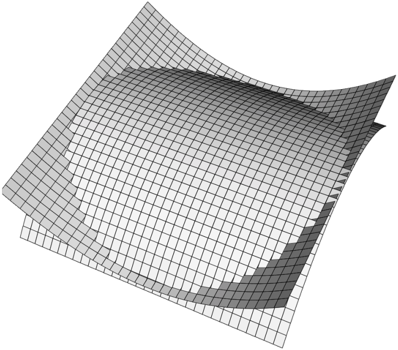
Results of `CtcInverse` provided in the Codac library

Comparison with affine arithmetic



Intersection of two lofted parabolas:

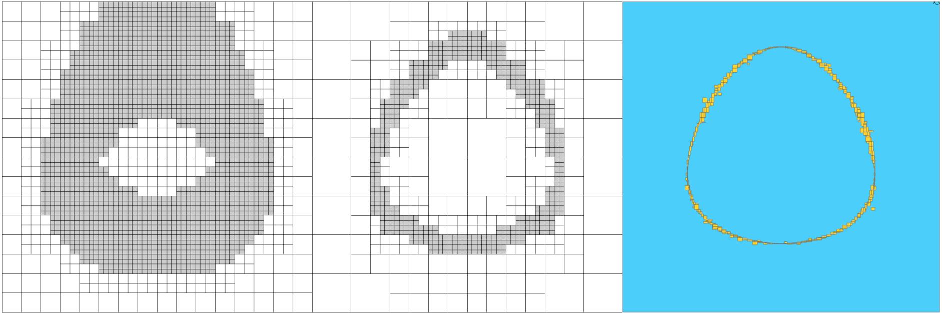
`L.H. De Figueiredo. Surface Intersection using Affine Arithmetic, Proceedings of Graphics Interface'96, 168-175. 1996`

$$\alpha(u) = \mathbf{a}_0(1-u)^2 + 2\mathbf{a}_1 u(1-u) + \mathbf{a}_2 u^2$$
$$\beta(u) = \mathbf{b}_0(1-u)^2 + 2\mathbf{b}_1 u(1-u) + \mathbf{b}_2 u^2$$
$$\mathbf{f}(u,v) = (1-v)\alpha(u) + v\beta(u)$$

Results of `CtcInverse` provided in the Codac library

Comparison with affine arithmetic

$$\mathbf{f}:\mathbb{R}^4\rightarrow\mathbb{R}^3$$



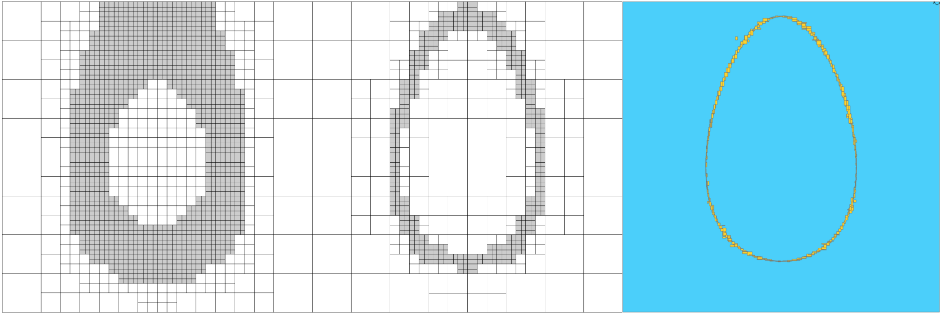
Results projected on the (u_1, v_1) space.
Left: natural evaluation – center: affine evaluation – right: `CtcInverse`

Using the same ϵ in the branch-and-band algorithm.

Results of `CtcInverse` provided in the Codac library

Comparison with affine arithmetic

$$\mathbf{f} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

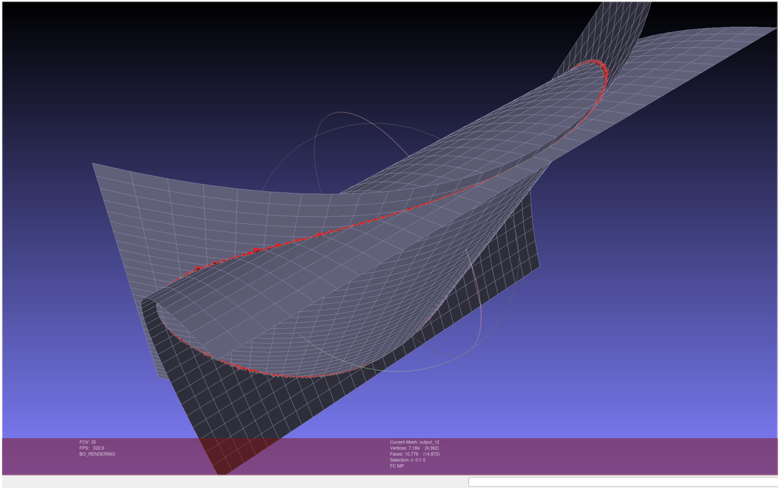


Results projected on the (u_2, v_2) space.
Left: natural evaluation – center: affine evaluation – right: `CtcInverse`

Using the same ϵ in the branch-and-band algorithm.

Results of `CtcInverse` provided in the Codac library

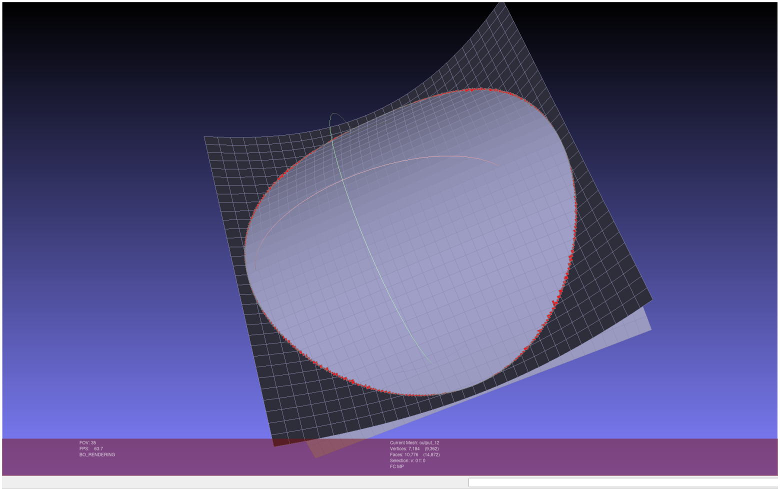
Comparison with affine arithmetic



Visualization of the approximation set (red) corresponding to the intersection of the parabolas.

Results of `CtcInverse` provided in the Codac library

Comparison with affine arithmetic



Visualization of the approximation set (red)
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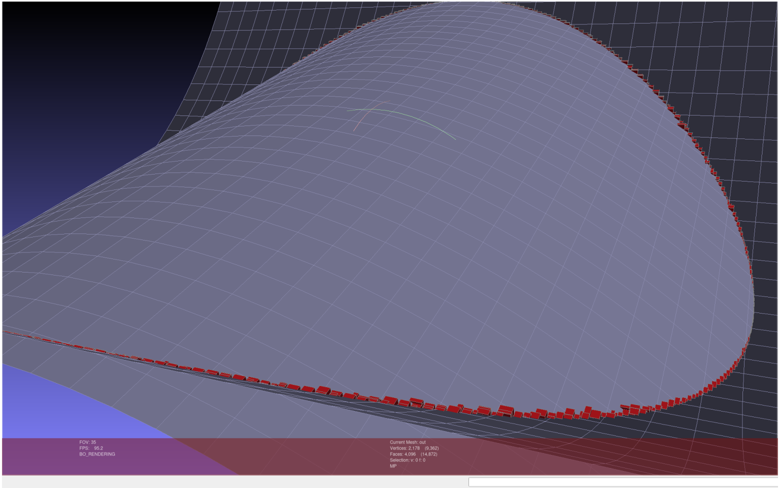
Comparison with affine arithmetic



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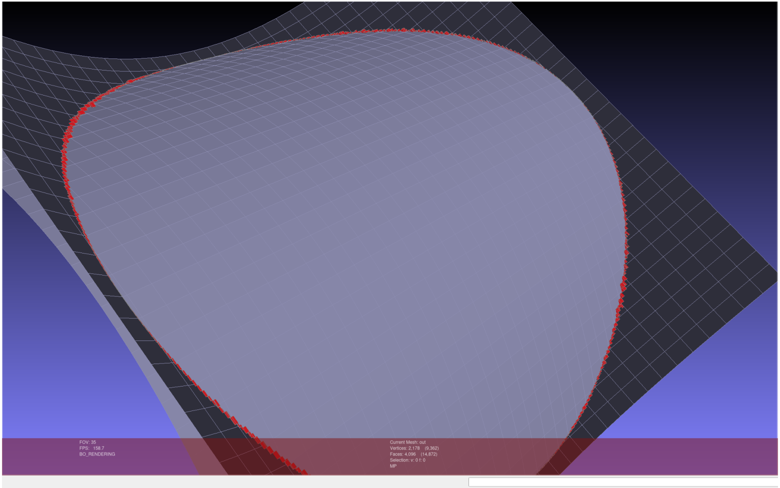
Results of `CtcInverse` provided in the Codac library

Comparison with affine arithmetic



Visualization of the approximation set (red) corresponding to the intersection of the parabolas.

Section 6

Conclusion

Conclusion

Contribution of this work

- **proposed algorithm involving IAD**
  - operator overloading:
  - implementation simple to maintain or to extend

Conclusion

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- **proposed algorithm involving IAD**
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  - simple to use for a non-advised user

Conclusion

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- **proposed algorithm involving IAD**
  - operator overloading:
  - implementation simple to maintain or to extend
- **embedded in the contractor framework**
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- **code available in C++, Python, Matlab**
  - and Linux, Windows, MacOS..
  - many thanks to Fabrice Le Bars for his help

Conclusion

Contribution of this work

- **proposed algorithm involving IAD**
  - operator overloading:
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- **embedded in the contractor framework**
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- **code available in C++, Python, Matlab**
  - and Linux, Windows, MacOS..
  - many thanks to Fabrice Le Bars for his help
- **comparison of a centered form algorithm with affine** arithmetic

Conclusion

Python example (Codac library)

```
from codac import *

x = VectorVar(3)
f = AnalyticFunction([x], vec(
  -sqr(x[2])+2*x[2]*sin(x[2]*x[0])+cos(x[2]*x[1]),
  2*x[2]*cos(x[2]*x[0])-sin(x[2]*x[1])
))

ctc = CtcInverse(f, [[0],[0]])
pave([[0,2],[2,4],[0,10]], ctc, 0.004)
```

Using pre-release Codac 2.0.0:

```
pip install codac --pre
```

http://codac.io

Conclusion

Matlab example (Codac library)

```
import py.codac4matlab.*

x = VectorVar(3);
f = AnalyticFunction({x}, vec( ...
  -sqr(x(3))+2*x(3)*sin(x(3)*x(1))+cos(x(3)*x(2)), ...
  2*x(3)*cos(x(3)*x(1))-sin(x(3)*x(2)) ...
));

ctc = CtcInverse(f, {0,0});
pave(IntervalVector({{0,2},{2,4},{0,10}}), ctc, 0.004);
```

Using pre-release Codac 2.0.0 dedicated to Matlab:

```
pip install codac4matlab --pre
```

http://codac.io

Conclusion

C++ example (Codac library)

```
#include <codac-core.h>

using namespace std;
using namespace codac2;

int main()
{
  VectorVar x(3);
  AnalyticFunction f({x}, vec(
    -sqr(x[2])+2*x[2]*sin(x[2]*x[0])+cos(x[2]*x[1]),
    2*x[2]*cos(x[2]*x[0])-sin(x[2]*x[1])
  ));

  CtcInverse_<IntervalVector> ctc(f, {0.,0.});
  pave(IntervalVector({{0,2},{2,4},{0,10}}), ctc, 0.004);
}
```

http://codac.io