Verifying loops in robot trajectories under uncertainties

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Section 1

Motivations

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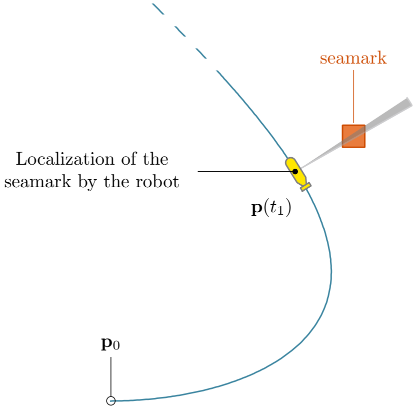
Simultaneous Localization and Mapping

- **come back** to a previous pose and **recognize** the environment
- problem: loop closure detection



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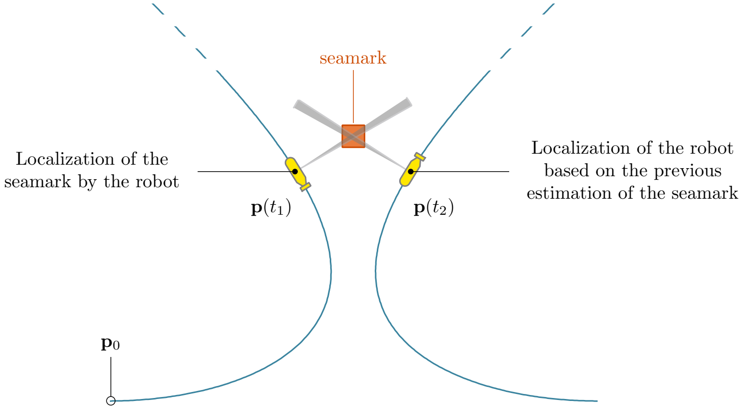
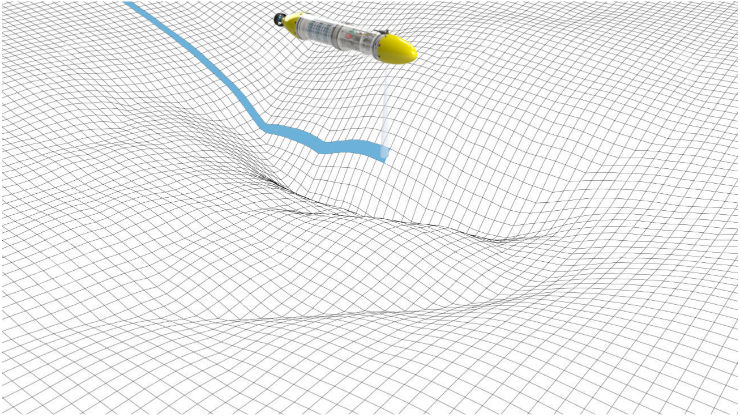


Motivations

Underwater robot localization



An underwater robot performing a loop during an exploration

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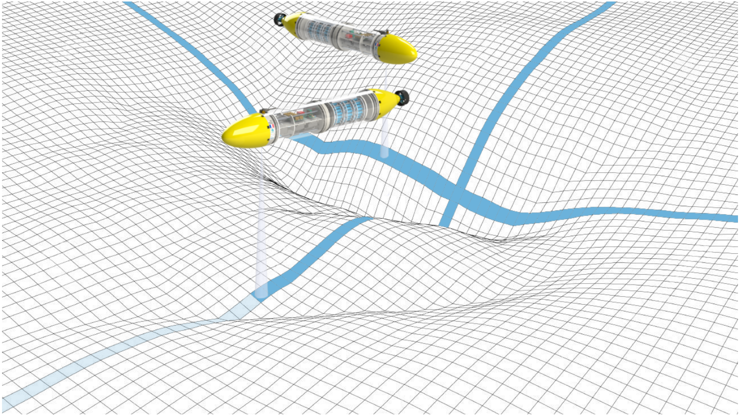

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Section 2

Looped trajectories

Looped trajectories

Definitions (Aubry, 2013)

- robot position: $\mathbf{p} = (x, y)^\intercal \in \mathbb{R}^2$
- 2D robot trajectory: $\mathbf{p}(t) : \mathbb{R} \to \mathbb{R}^2$, $t \in [t_0, t_f]$
- looped trajectory $\Leftrightarrow$ trajectory that crosses itself
  - $\mathbf{p}(t_1) = \mathbf{p}(t_2)$, $t_1 \neq t_2$
  - 1 loop $\Leftrightarrow$ 1 t-pair (t_1, t_2)

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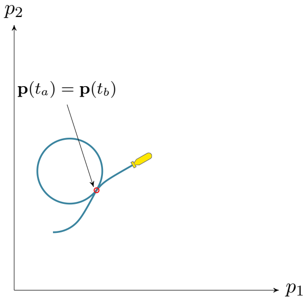

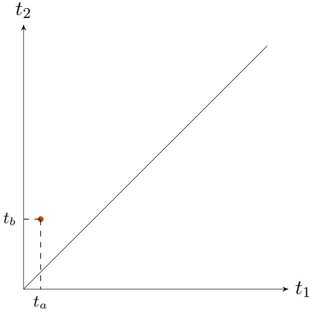

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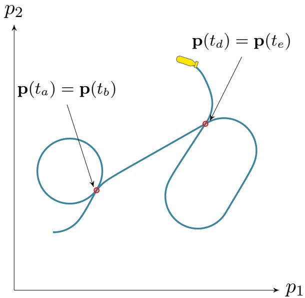

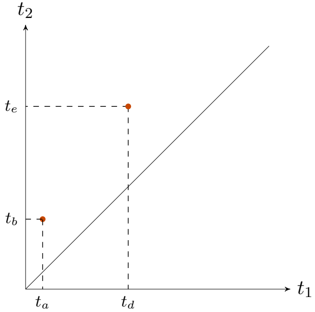

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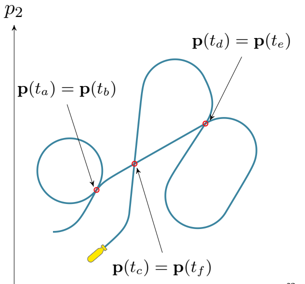
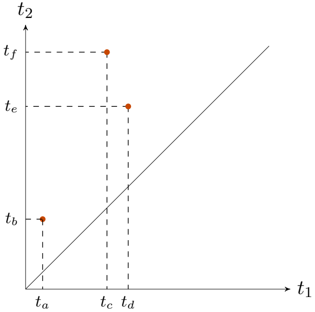
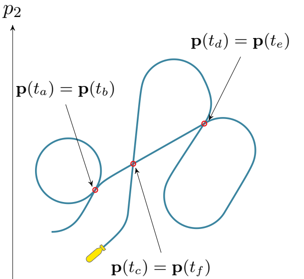


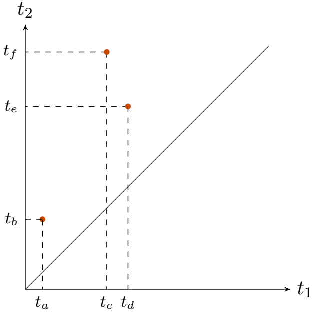


Looped trajectories

Definitions (Aubry, 2013)

- t-plane $\Leftrightarrow$ all feasible t-pairs $= [t_0, t_f]^2$
- *loop set* $\mathbb{T}^*$:
  - $\mathbb{T}^* = \left\{(t_1, t_2) \in [t_0, t_f]^2 \mid \mathbf{p}(t_1) = \mathbf{p}(t_2), t_1 < t_2\right\}$
- *loop set* of below example:
  - $\mathbb{T}^* = \{(t_a, t_b), (t_c, t_f), (t_d, t_e)\}$





Looped trajectories

Computing loops from robot sensors

Context: robot trajectory $\mathbf{p}(t)$ cannot be directly sensed.
Computation from speed measurements:

$$\mathbf{p}(t) = \int_{t_0}^{t} \mathbf{v}(\tau)d\tau + \mathbf{p}_0, \tag{1}$$

with $\mathbf{v}(t) \in \mathbb{R}^2$: robot velocity vector at time $t \in [t_0, t_f]$.

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Loop-set from velocity:

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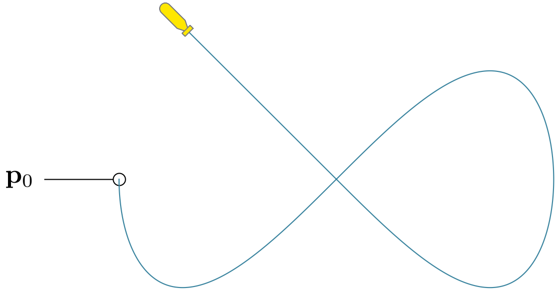
Section 3

Uncertain trajectories

Uncertain trajectories

Set-membership approach

Actual trajectory: $\mathbf{p}(t) = \int_{t_0}^{t} \mathbf{v}(\tau) d\tau + \mathbf{p}_0$



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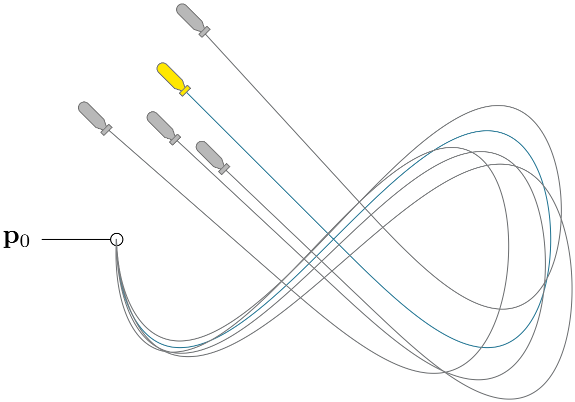
Drifting trajectory: $\mathbf{p}_e(t) = \int_{t_0}^{t} \big(\mathbf{v}(\tau) + \epsilon(\tau)\big)d\tau + \mathbf{p}_0$



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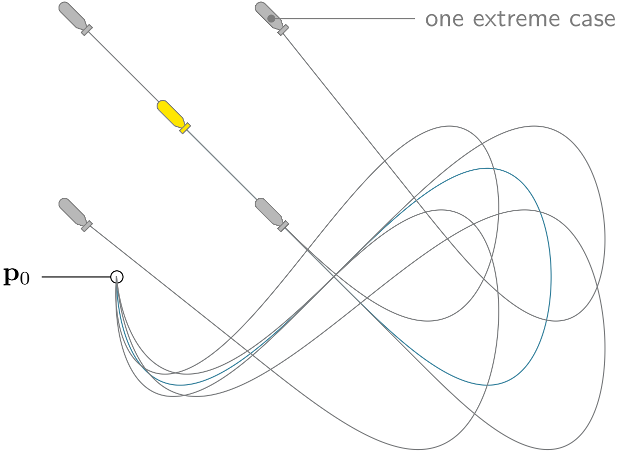
Approach: consider worst cases by defining bounded solutions



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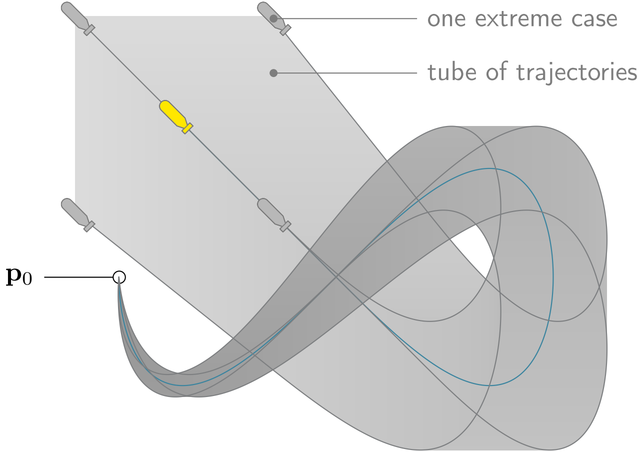
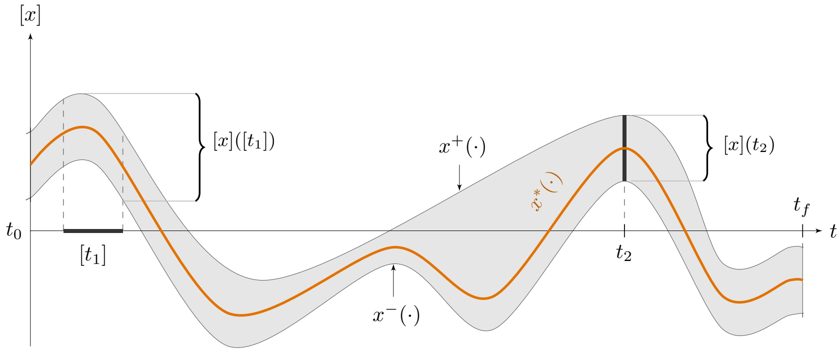


Uncertain trajectories

Tubes

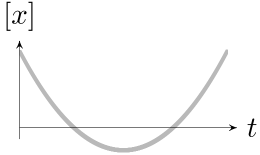
Tube $[x](\cdot)$: interval of trajectories $[x^-(\cdot), x^+(\cdot)]$
such that $\forall t \in \mathbb{R},\ x^-(t) \leqslant x^+(t)$

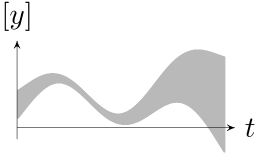


Tube $[x](\cdot)$ enclosing an uncertain trajectory $x^*(\cdot)$

Uncertain trajectories

Tubes arithmetic





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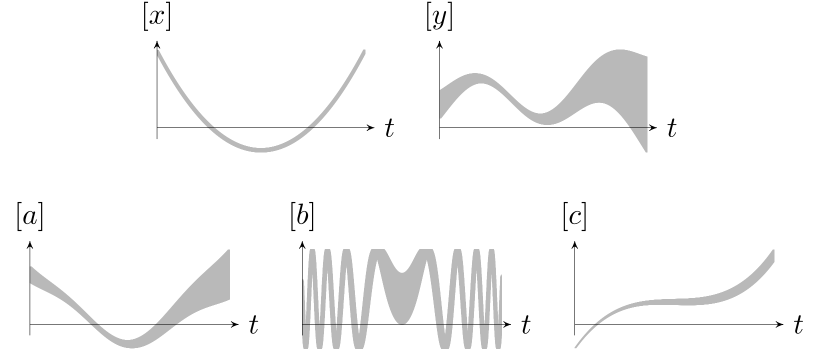


$$[a](\cdot) = [x](\cdot) + [y](\cdot)$$

$$[b](\cdot) = \sin\big([x](\cdot)\big)$$

$$[c](\cdot) = \int_0 [x](\tau)d\tau$$

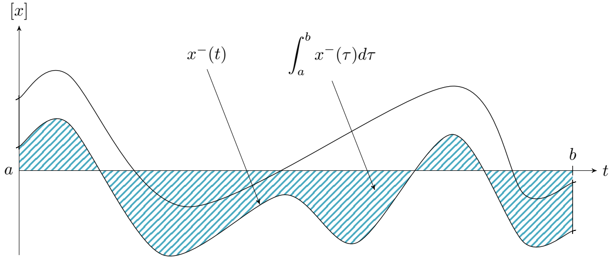
Uncertain trajectories

Integral of tubes

Definition: the integral of a tube $[x](\cdot) = [x^-, x^+]$ is an interval:

$$\int_a^b [x](\tau)d\tau = \left\{ \int_a^b x(\tau)d\tau \mid x(\cdot) \in [x](\cdot) \right\} = \left[\int_a^b x^-(\tau)d\tau, \int_a^b x^+(\tau)d\tau \right]$$

[Aubry2013]



blue area: lower bound of the tube's integral

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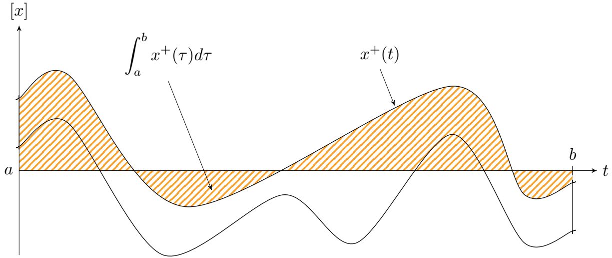

orange area: upper bound of the tube's integral

Section 4

Loop detection

Loop detection

Bounded-error context

Actual loop-set $\mathbb{T}^*$ (error free):

$$\mathbb{T}^* = \left\{ (t_1, t_2) \mid \int_{t_1}^{t_2} \mathbf{v}^*(\tau) d\tau = \mathbf{0} \right\} \tag{4}$$

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Bounded-error context, assuming $\mathbf{v}^*(\cdot) \in [\mathbf{v}](\cdot)$:

$$\mathbb{T} = \left\{ (t_1, t_2) \mid \exists \mathbf{v}(\cdot) \in [\mathbf{v}](\cdot) \ , \int_{t_1}^{t_2} \mathbf{v}(\tau) d\tau = \mathbf{0} \right\} \tag{5}$$

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Set-membership approach:

$$\mathbb{T}^* \subset \mathbb{T} \subset [t_0, t_f]^2 \tag{6}$$

Loop detection

Inclusion function

Simplification:
defining the actual but unknown function $\mathbf{f}^* : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}^*(t_1, t_2) = \int_{t_1}^{t_2} \mathbf{v}^*(\tau) d\tau \qquad (7)$$

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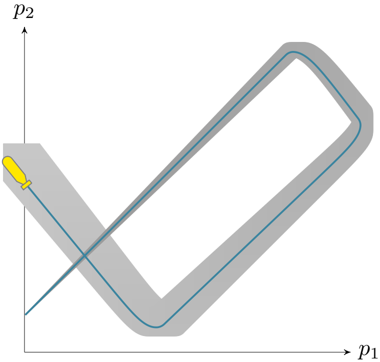
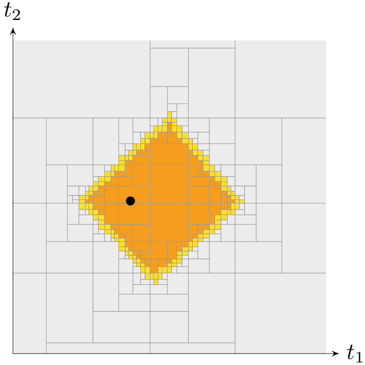
Assessed knowledge:
$[\mathbf{f}] : \mathbb{R}^2 \to \mathbb{IR}^2$ is an *interval function* of $\mathbf{f}^*$:

$$\mathbf{f}^*(t_1, t_2) \in [\mathbf{f}](t_1, t_2) = \int_{t_1}^{t_2} [\mathbf{v}](\tau)d\tau \tag{8}$$

Loop detection

Reliable approximation of a loop set





Undeniable looped trajectory

Loop detection

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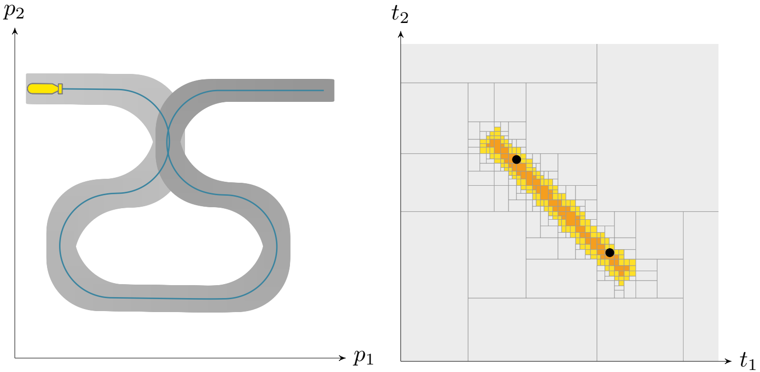

Doubtful looped trajectory

Loop detection

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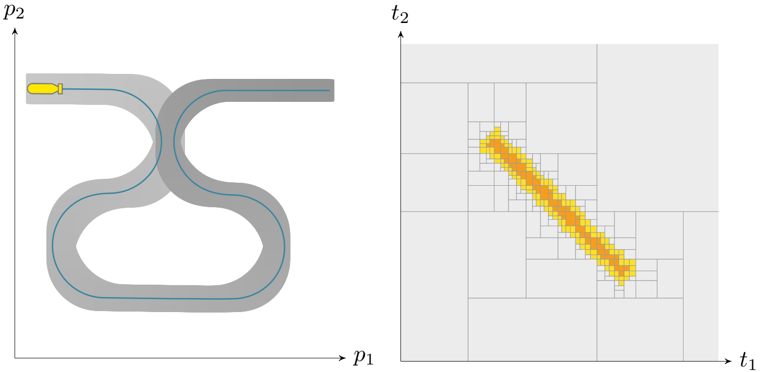

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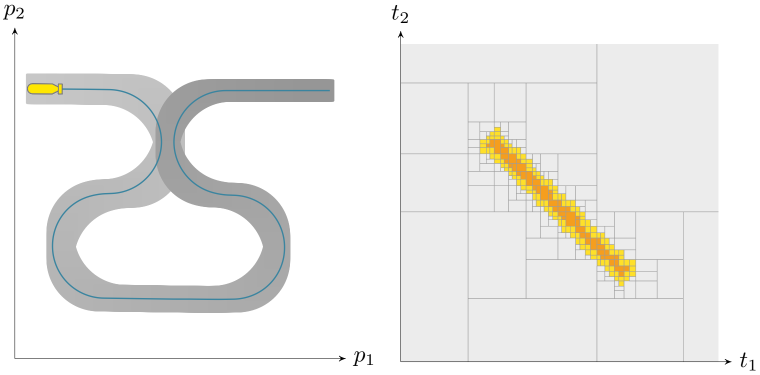

Doubtful looped trajectory

$$\forall \mathbf{f} \in [\mathbf{f}], \exists \mathbf{t} \in \mathbb{T}_i \mid \mathbf{f}(\mathbf{t}) = \mathbf{0} \implies \underbrace{\exists \mathbf{t} \in \mathbb{T}_i \mid \mathbf{f}^*(\mathbf{t}) = \mathbf{0}}_{\text{loop existence proof}} \qquad (9)$$

Section 5

Topological degree for zero verification

Topological degree for zero verification

Problem statement

Statement:

- known inclusion function $[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ of the unknown function $\mathbf{f}^* : \mathbb{R}^n \to \mathbb{R}^m$
- possibly in the form of an algorithm for computing $[\mathbf{f}]([\mathbf{t}])$
- $n = m = 2$
- need to isolate and verify zeros of $\mathbf{f}^*$

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Zero verification:

1. if $\mathbf{0} \notin [\mathbf{f}]([\mathbf{t}])$ for some box $[\mathbf{t}]$, then $\mathbf{f}^*$ has no zero on $[\mathbf{t}]$
2. harder to verify the *existence* of zero inside a region
   - if $\mathbf{0} \in [\mathbf{f}]([\mathbf{t}])$, we cannot disprove $\mathbf{f}^*(\mathbf{t}) = \mathbf{0}$ for some $\mathbf{t}$
   - but it is also not obvious how to prove the existence of such $\mathbf{t}$

Topological degree for zero verification

Powerful topological degree

Topological degree $\deg(\mathbf{f}^*, \Omega)$:

- unique integer assigned to $\mathbf{f}^*$ and a compact set $\Omega \subset \mathbb{R}^n$ such that $\mathbf{f}^*(\mathbf{t}) \neq \mathbf{0}$ for all $\mathbf{t} \in \partial\Omega$

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Most important property of it:

$$\deg(\mathbf{f}^*, \Omega) \neq 0 \quad \Longrightarrow \quad \exists \mathbf{t} \in \Omega \mid \mathbf{f}^*(\mathbf{t}) = \mathbf{0} \qquad (10)$$

Topological degree theory and applications
Y. J. Cho, Y. Q. Chen *Mathematical Analysis and Applications*, 2006

Degree theory in analysis and applications
I. Fonseca, W. Gangbo *Oxford lecture series*, 1995

A set of axioms for the degree of a tangent vector field on differentiable manifolds
M. Furi, M. P. Pera, M. Spadini *Fixed Point Theory and Applications*, 2010

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Assets of topological degree:

- can be computed in case where only an inclusion function $[\mathbf{f}]$ of $\mathbf{f}^*$ is given

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- useful to count the number of $\mathbf{0}$

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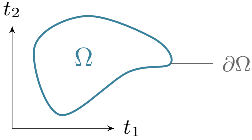

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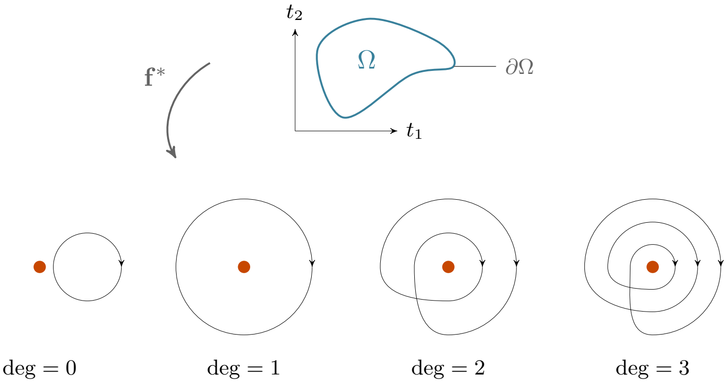
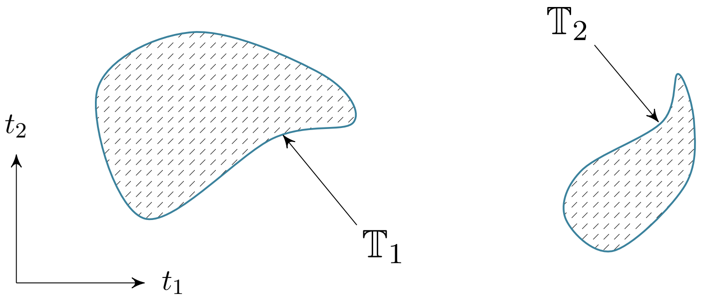


Topological degree for zero verification

Outer approximation of a set $\mathbb{T}$ with SIVIA

Consider $\mathbb{T} \subset \mathbb{R}^n$ in which we want to find zeros of $\mathbf{f}^*$.



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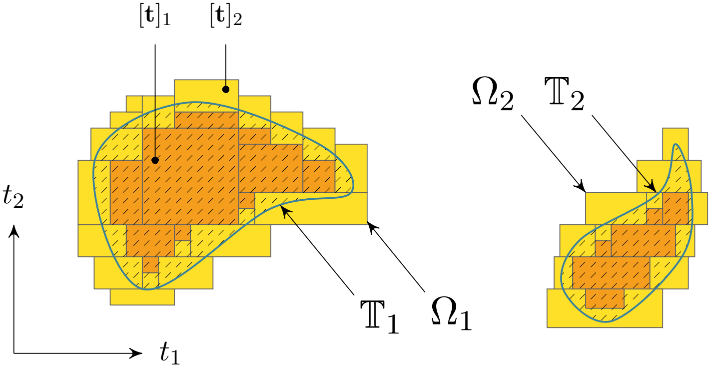

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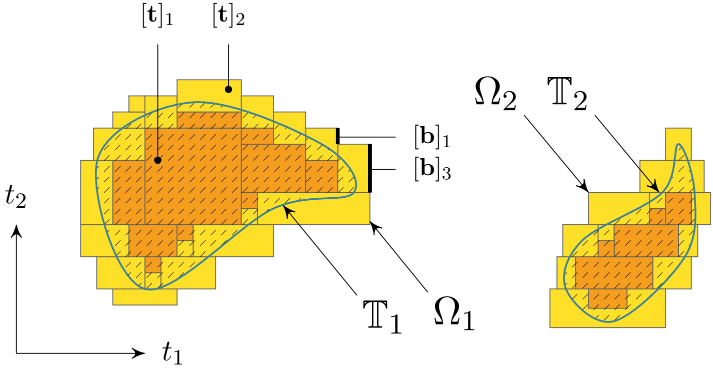

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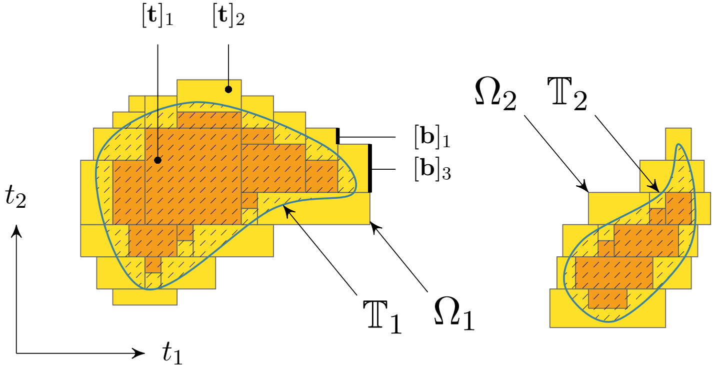
Outer approximation of a set $\mathbb{T}$ with SIVIA

Consider $\mathbb{T} \subset \mathbb{R}^n$ in which we want to find zeros of $\mathbf{f}^*$.



Outer set has the properties required for Ω: $\mathbf{f}^*(\mathbf{t}) \neq \mathbf{0},\ \forall \mathbf{t} \in \partial\Omega$

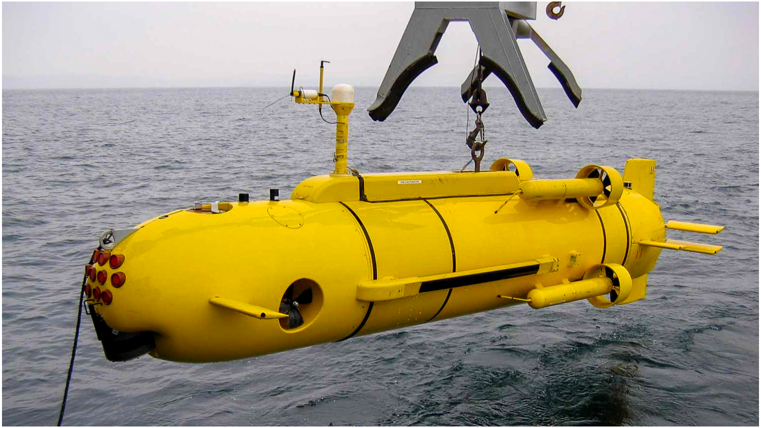
Section 6

Application

Application

Redermor mission

2 hours experimental mission in Brittany (France)



The *Redermor* Autonomous Underwater Vehicle (AUV)

Application

Reliable approximation of absolute speed $\mathbf{v}^*(\cdot)$

Robot sensors for **absolute speed computation**:

- velocity sensor (DVL)
- inertial measurement unit

Application

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Uncertainties:

- datasheets $\implies$ standard deviation σ for each sensor

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Reliable approximation of absolute speed $\mathbf{v}^*(\cdot)$

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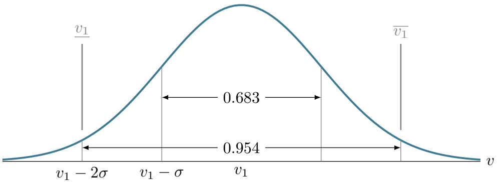
- 95% confidence rate: $v_1^* \in [v_1] = [v_1 - 2\sigma, v_1 + 2\sigma]$



Application

Reliable approximation of absolute speed $\mathbf{v}^*(\cdot)$

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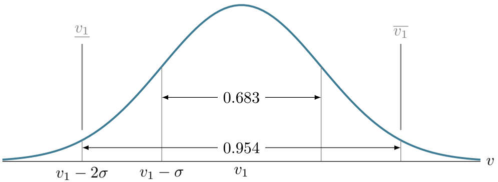
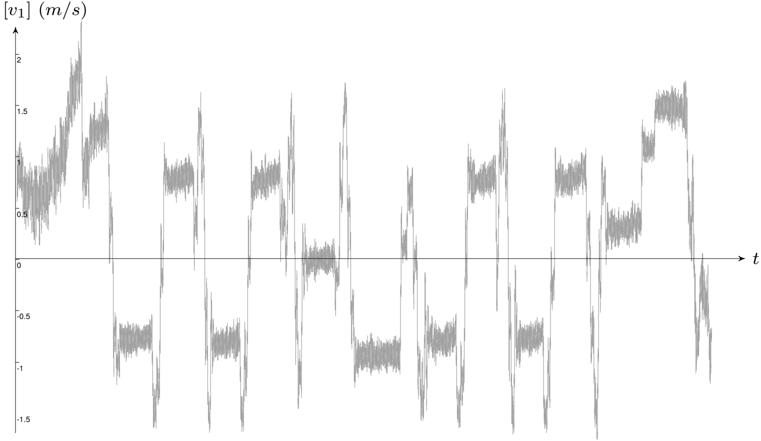


- uncertainties propagated thanks to interval arithmetic

Application

Reliable approximation of absolute speed $\mathbf{v}^*(\cdot)$

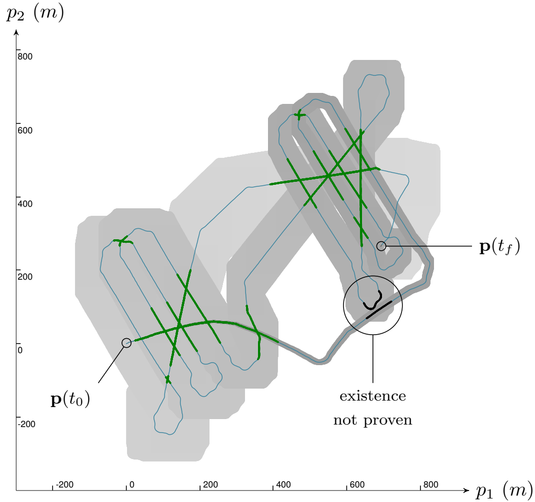
Obtained tube $[\mathbf{v}](\cdot)$:



East speed velocity tube $[v_1](\cdot)$

Application

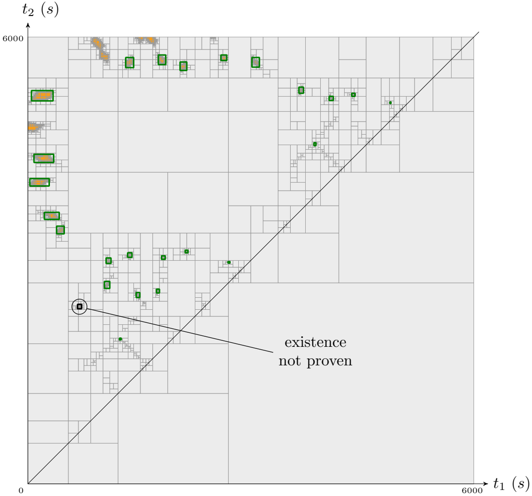
Guaranteed computation of the trajectory



2d trace of *Redermor* AUV

Application

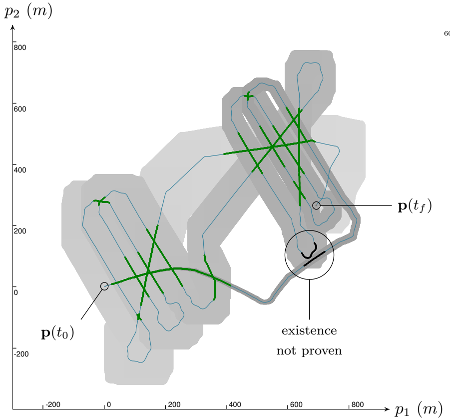
t-plane of the mission: $\mathbb{T} = \{(t_1, t_2) \mid \mathbf{0} \in [\mathbf{f}](t_1, t_2), t_1 < t_2\}$

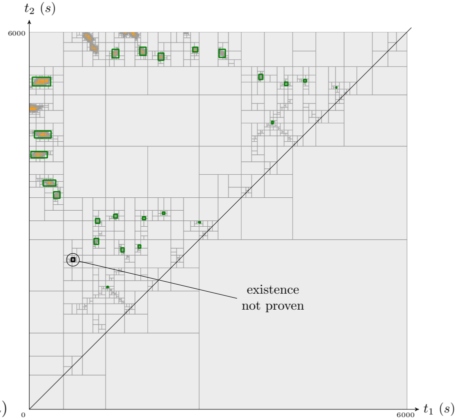


t-plane corresponding to *Redermor*'s mission

Application

Overview and results





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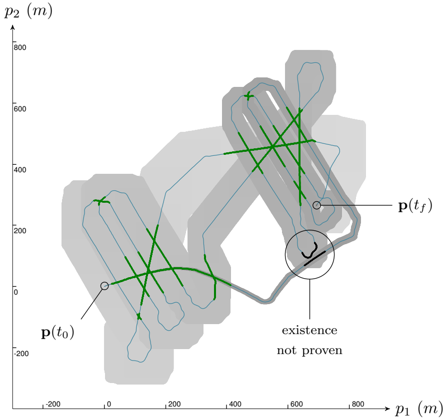
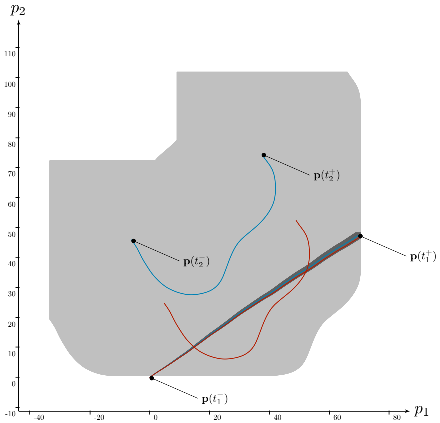
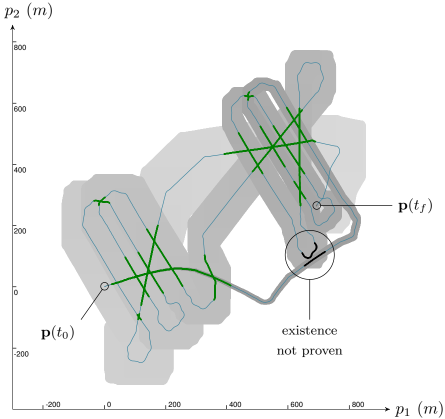




Application

Overview and results



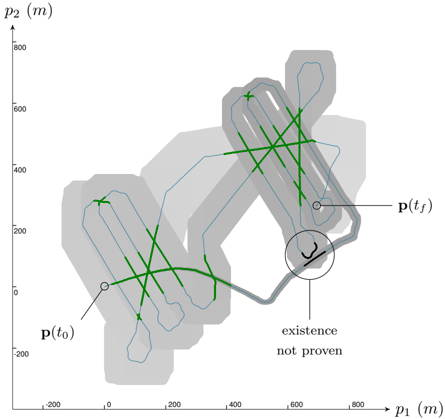
Loop proof number

Without uncertainties:

$$\lambda^* = \#\left\{\mathbf{t} \mid \mathbf{f}^*(\mathbf{t}) = \mathbf{0}, t_1 < t_2\right\}$$

Application

Overview and results



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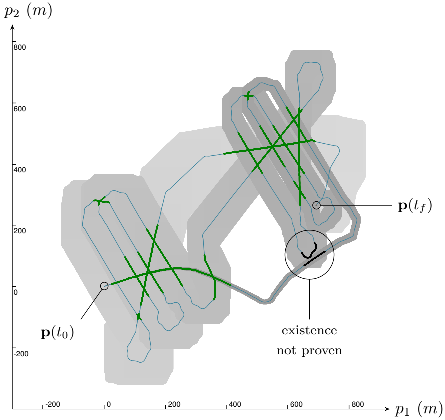
Results:

Newton operator test: $\lambda_{\mathcal{N}} = 14$

Application

Overview and results



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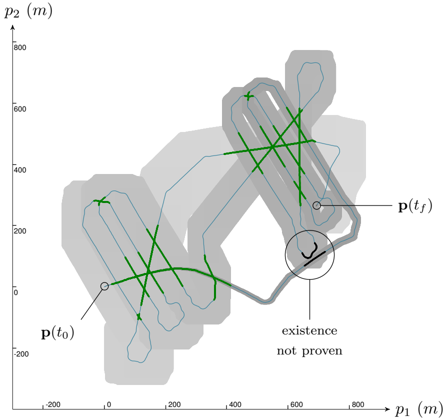
Topological degree test: $\lambda_{\mathcal{T}} = 24$

Application

Overview and results



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Results:

Newton operator test: $\lambda_{\mathcal{N}} = 14$

Topological degree test: $\lambda_{\mathcal{T}} = 24$

Truth: $\lambda^* = 24$

Conclusion

Loop proof ⇔ **verified existence of a 0** of an uncertain function:

- situation where the exact values of the function are not known
- have to deal with a reliable approximation of it

Conclusion

Loop proof $\Leftrightarrow$ **verified existence of a 0** of an uncertain function:

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Topological degree theory:

- well suited in this case
- applied in a 2d context

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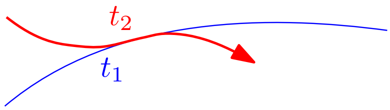

Conclusion

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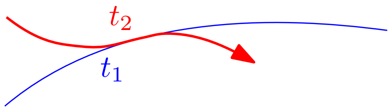
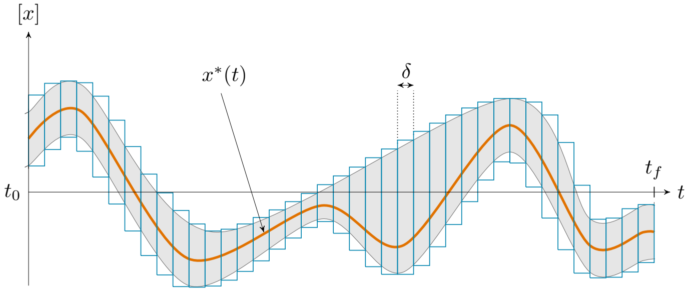


Proving the existence of loops in robot trajectories
S. Rohou, P. Franek, C. Aubry, L. Jaulin *The International Journal of Robotics Research*, 2018

Available software

An open-source C++ (Python) library based on IBEX and providing tools for constraint programming over dynamical systems.

- `Tube`, `TubeVector`, ...
- contractors (delays, differential eq., time uncertainties...)
- robotic tools and applications



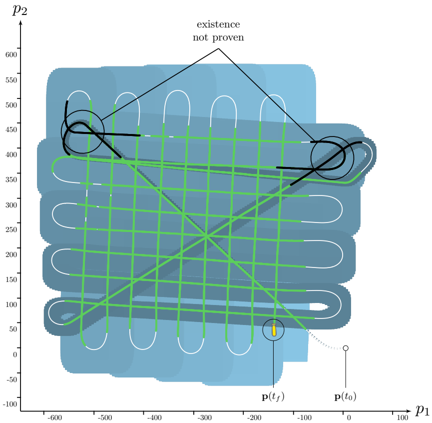
http://www.simon-rohou.fr/research/tubex-lib/

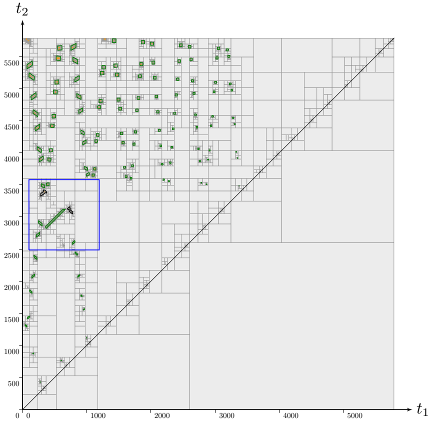
Section 9

Appendix

Appendix

Another experiment (*Daurade* AUV)





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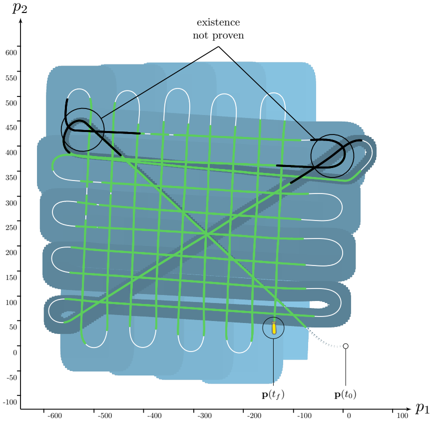

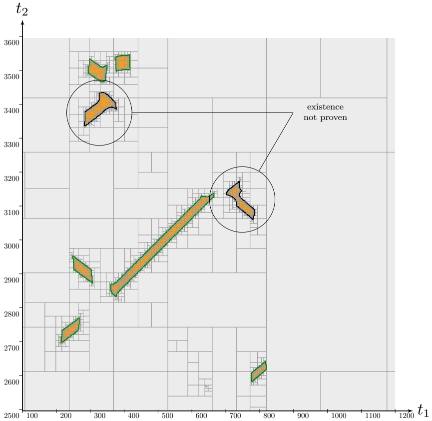

Appendix

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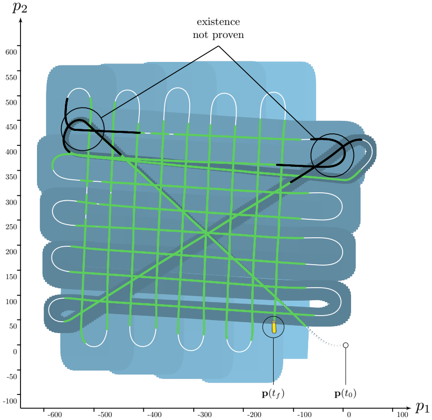
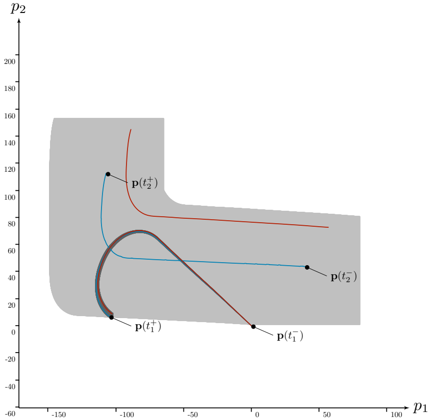


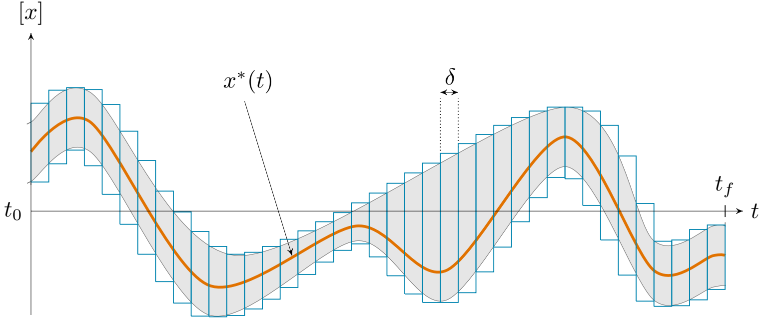


Appendix

Tubes: computer representation

Implementation **enclosing** $[x^-(\cdot), x^+(\cdot)]$ inside an interval of step functions $[\underline{x^-}(\cdot), \overline{x^+}(\cdot)]$ such that:

$$\forall t \in \mathbb{R},\ \underline{x^-}(t) \leqslant x^-(t) \leqslant x^+(t) \leqslant \overline{x^+}(t)$$



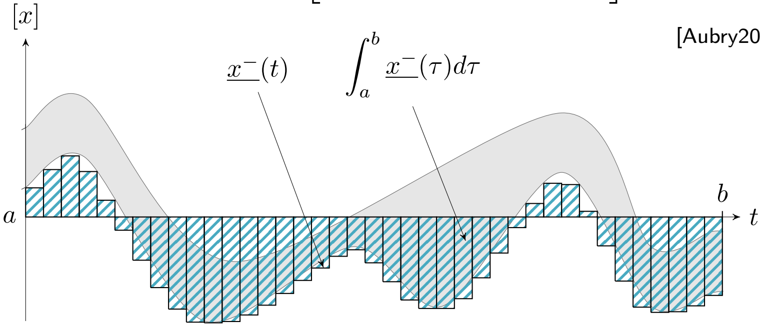
tube implementation with a set of boxes – this outer representation adds pessimism but enables guaranteed and simple computations

Appendix

Tubes integral: implementation

Outer approximation of the integral computed by:

$$\int_a^b [x](\tau)d\tau \subset \left[\int_a^b \underline{x^-}(\tau)d\tau, \int_a^b \overline{x^+}(\tau)d\tau\right]$$



blue area: outer approximation of the lower bound of the tube's integral