

A Reliable Way To Compute Robot Trajectories

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The
University
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Sheffield.



Section 1

Introduction

Introduction

Mobile Robotics: formalization

Robot localization = state estimation problem.

Classically, we have:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & \text{(navigation)} \\ y(t) = g(\mathbf{x}(t)) & \text{(measurements)} \end{cases}$$

Where:

- ▶ $\mathbf{x} \in \mathbb{R}^n$ is the state vector (position, bearing, ...)
- ▶ $\mathbf{u} \in \mathbb{R}^m$ is the input vector (command)
- ▶ $y \in \mathbb{R}$ is some exteroceptive measurement (e.g. bathymetry)
- ▶ $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the *evolution* function
- ▶ $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *observation* function

Introduction

Constraint Network based on trajectories

A problem involving constraints is classically presented with a **Constraint Network**:

[Mackworth1977], [Chabert2009]

{
 Variables:
 Constraints:

 Domains:

Introduction

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{ **Variables:** $x(\cdot), u(\cdot), y$
Constraints:
Domains:

Our approach consists in using:

- ▶ **trajectories** as variables: $x(\cdot), u(\cdot)$

Introduction

Constraint Network based on trajectories

A problem involving constraints is classically presented with a **Constraint Network**:

[Mackworth1977], [Chabert2009]

Variables: $\mathbf{x}(\cdot), \mathbf{u}(\cdot), y$
Constraints:
Domains: $[\mathbf{x}](\cdot), [\mathbf{u}](\cdot), [y]$

Our approach consists in using:

- ▶ **trajectories** as variables: $\mathbf{x}(\cdot), \mathbf{u}(\cdot)$
- ▶ **tubes** as domains: $\mathbf{x}(\cdot) \in [\mathbf{x}](\cdot), \mathbf{u}(\cdot) \in [\mathbf{u}](\cdot)$

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[Mackworth1977], [Chabert2009]

$$\left\{ \begin{array}{l} \mathbf{Variables:} \quad \mathbf{x}(\cdot), \mathbf{u}(\cdot), y \\ \mathbf{Constraints:} \\ \quad - \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad \text{(evolution equation)} \\ \quad - \quad y(t) = g(\mathbf{x}(t)) \quad \text{(observation equation)} \\ \mathbf{Domains:} \quad [\mathbf{x}](\cdot), [\mathbf{u}](\cdot), [y] \end{array} \right.$$

Our approach consists in using:

- ▶ **trajectories** as variables: $\mathbf{x}(\cdot), \mathbf{u}(\cdot)$
- ▶ **tubes** as domains: $\mathbf{x}(\cdot) \in [\mathbf{x}](\cdot), \mathbf{u}(\cdot) \in [\mathbf{u}](\cdot)$

Now we can consider constraints defined with **differential equations** or any other **non-linear operation** on trajectories.

Section 2

Tubes: Envelopes of Trajectories

Tubes: Envelopes of Trajectories

Definition

Tube $[x](\cdot)$: interval of functions $[x^-, x^+]$ such that $\forall t \in \mathbb{R}, x^-(t) \leq x^+(t)$

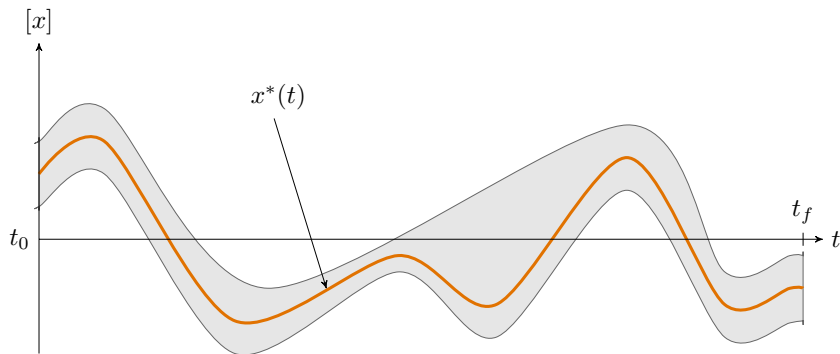


Figure: tube $[x](\cdot)$ enclosing an uncertain trajectory $x^*(\cdot)$

Tubes: Envelopes of Trajectories

Tubes arithmetic

Example:

Tube arithmetic makes it possible to compute the following tubes:

$$[a](\cdot) = [x](\cdot) + [y](\cdot)$$

$$[b](\cdot) = \sin([x](\cdot))$$

$$[c](\cdot) = \int_0^{\cdot} [x](\tau) d\tau$$

Definition:

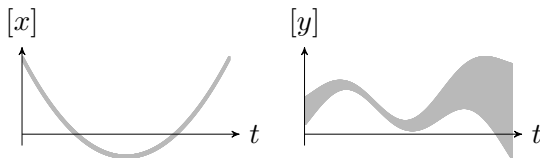
If f is an elementary function such as \sin , \cos , \dots ,

$f([x](\cdot))$ is the smallest tube containing all feasible values for

$f(x(\cdot))$, $x(\cdot) \in [x](\cdot)$.

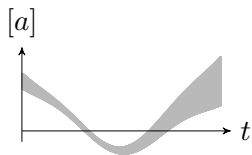
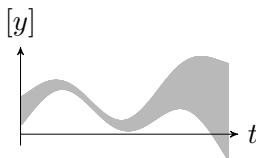
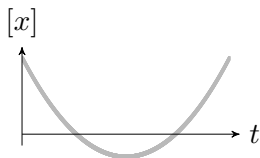
Tubes: Envelopes of Trajectories

Tubes arithmetic: example



Tubes: Envelopes of Trajectories

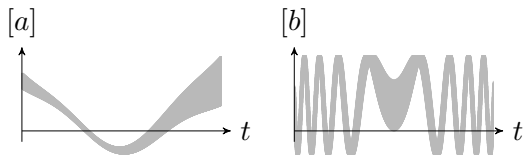
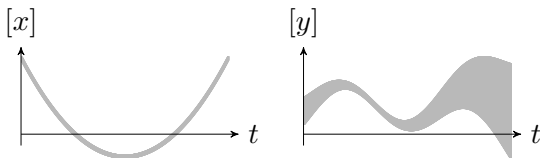
Tubes arithmetic: example



$$a(\cdot) = x(\cdot) + y(\cdot)$$

Tubes: Envelopes of Trajectories

Tubes arithmetic: example

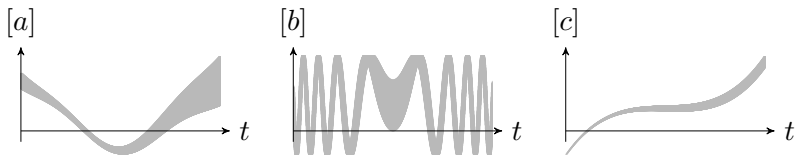
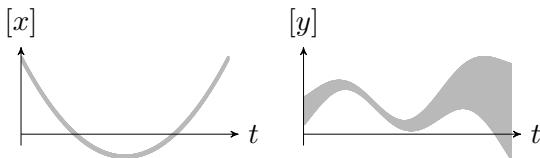


$$a(\cdot) = x(\cdot) + y(\cdot)$$

$$b(\cdot) = \sin(x(\cdot))$$

Tubes: Envelopes of Trajectories

Tubes arithmetic: example



$$a(\cdot) = x(\cdot) + y(\cdot)$$

$$b(\cdot) = \sin(x(\cdot))$$

$$c(\cdot) = \int_0^{\cdot} x(\tau) d\tau$$

Section 3

Tube contractors

Tube contractors

Tubes: arithmetic and contractors

Example of constraints on trajectories:

$$a(\cdot) = x(\cdot) + y(\cdot)$$

$$b(\cdot) = \sin(x(\cdot))$$

$$c(\cdot) = \int_0^{\cdot} x(\tau) d\tau$$

Tubes $[a](\cdot)$, $[b](\cdot)$, $[c](\cdot)$ respectively contain solutions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$.

Tube contractors

Tubes: arithmetic and contractors

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Tubes $[a](\cdot)$, $[b](\cdot)$, $[c](\cdot)$ respectively contain solutions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$.

Contractors for tubes:

For each primitive constraint, tubes from both sides of the equation are contracted without removing any feasible solution.

Tube contractors

Tubes: minimal and non-minimal contractors

Example:

Minimal contractor \mathcal{C}_+ associated with the constraint

$$a(\cdot) = x(\cdot) + y(\cdot):$$

$$\begin{pmatrix} [a] (\cdot) \\ [x] (\cdot) \\ [y] (\cdot) \end{pmatrix} \mapsto \begin{pmatrix} [a] (\cdot) \cap ([x] (\cdot) + [y] (\cdot)) \\ [x] (\cdot) \cap ([a] (\cdot) - [y] (\cdot)) \\ [y] (\cdot) \cap ([a] (\cdot) - [x] (\cdot)) \end{pmatrix}$$

Tube contractors

Tubes: minimal and non-minimal contractors

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Minimal contractor \mathcal{C}_+ associated with the constraint

$$a(\cdot) = x(\cdot) + y(\cdot):$$

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Example:

Non-minimal contractor \mathcal{C}_{\int_0} associated with the constraint

$$c(\cdot) = \int_0 x(\tau) d\tau:$$

$$\begin{pmatrix} [x](\cdot) \\ [c](\cdot) \end{pmatrix} \mapsto \begin{pmatrix} [x](\cdot) \\ [c](\cdot) \cap \int_0 [x](\tau) d\tau \end{pmatrix}$$

Tube contractors

Contractors for state estimation

State equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & \text{(evolution)} \\ y(t) = g(\mathbf{x}(t)) & \text{(observation)} \end{cases}$$

Need for a **differential** contractor $\mathcal{C}_{\frac{d}{dt}}$ and a **probe** contractor \mathcal{C}_{obs} .

Tube contractors

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Need for a **differential** contractor $\mathcal{C}_{\frac{d}{dt}}$ and a **probe** contractor \mathcal{C}_{obs} .

- ▶ $\mathcal{C}_{\frac{d}{dt}}$: related to the constraint $\dot{\mathbf{x}} = \mathbf{v}$
 $\mathbf{x}(\cdot) \in [\mathbf{x}](\cdot), \mathbf{v}(\cdot) \in [\mathbf{v}](\cdot)$

Tube contractors

Contractors for state estimation

State equations:

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 $\mathbf{x}(\cdot) \in [\mathbf{x}](\cdot), \mathbf{v}(\cdot) \in [\mathbf{v}](\cdot)$
- ▶ \mathcal{C}_{obs} : related to the constraint $\mathbf{x}(t) = \mathbf{z}$
 $t \in [t], \mathbf{z} \in [\mathbf{z}], \mathbf{x}(\cdot) \in [\mathbf{x}](\cdot)$

Tube contractors

Contractors for state estimation

\mathcal{C}_{obs} : contraction based on the observation $[z_1]$ made at time $[t_1]$.

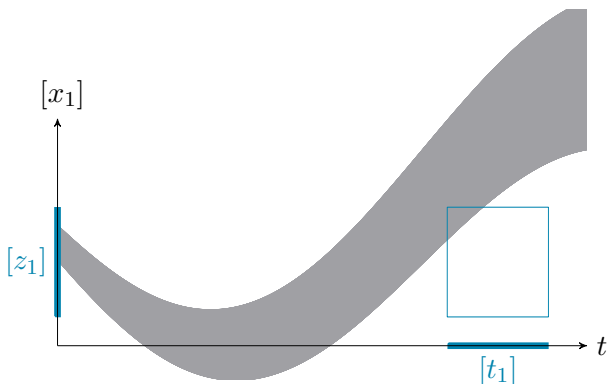


Figure: tube $[x_1](\cdot)$ before contraction

Tube contractors

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\mathcal{C}_{obs} : contraction based on the observation $[z_1]$ made at time $[t_1]$.

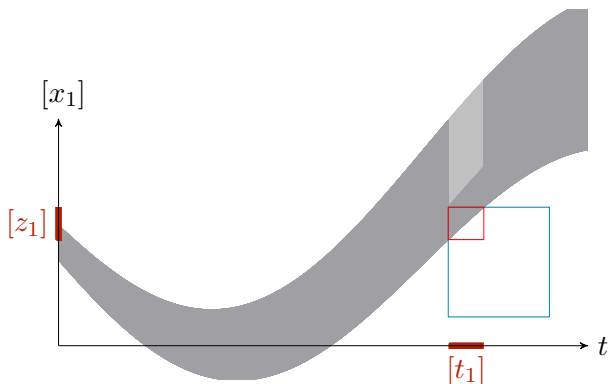


Figure: contraction of tube $[x_1](\cdot)$ and both $[z_1]$ and $[t_1]$

Tube contractors

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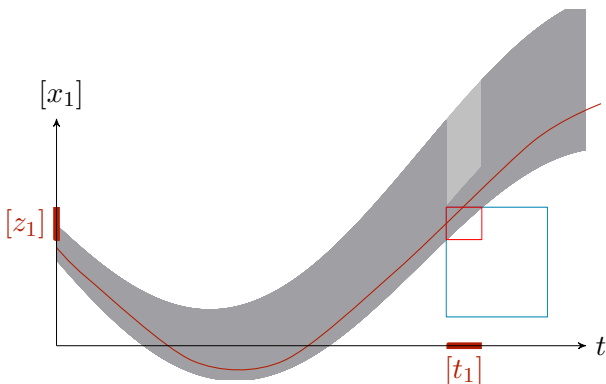


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Tube contractors

Contractors for state estimation

$\mathcal{C}_{\frac{d}{dt}}$: contraction based on the differential constraint:

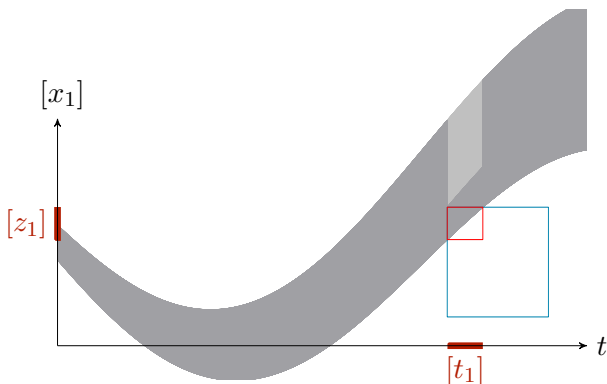


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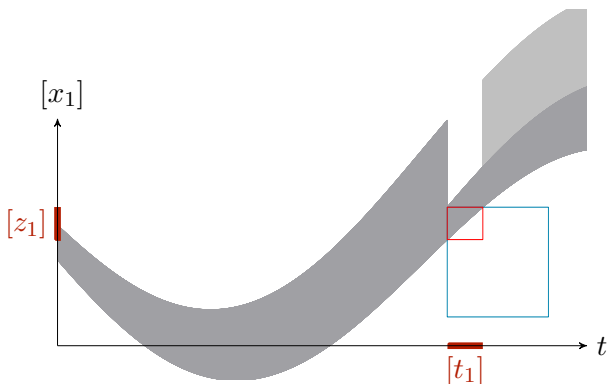


Figure: tube contraction in forward

Tube contractors

Contractors for state estimation

$\mathcal{C}_{\frac{d}{dt}}$: contraction based on the differential constraint:

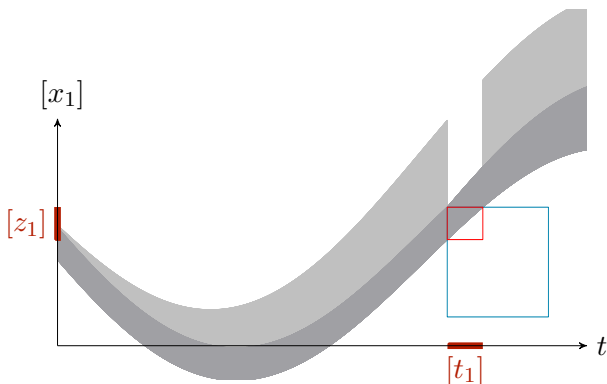


Figure: tube contraction in forward/backward

Section 4

Simulation of Mobile Robots

Simulation of Mobile Robots

Dead-reckoning

Considering a car \mathcal{R} which state is $\mathbf{x} = (x, y, \theta)^\top$.

$$\mathcal{R} \begin{cases} \dot{x}(t) &= v \cdot \cos(\theta) \\ \dot{y}(t) &= v \cdot \sin(\theta) \\ \dot{\theta}(t) &= u(t) \end{cases}$$

Simulation of Mobile Robots

Dead-reckoning

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- ▶ car's speed: $v = 10m.s^{-1}$
- ▶ $\mathbf{x}_0 \in [-1, 1] \times [-1, 1] \times [\frac{-6\pi}{5} - 0.002, \frac{-6\pi}{5} + 0.002]$
- ▶ $u(t) \in -\cos\left(\frac{t+33}{5}\right) + [-0.02, 0.02]$

Simulation of Mobile Robots

Dead-reckoning

Considering a car \mathcal{R} which state is $\mathbf{x} = (x, y, \theta)^\top$.

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- ▶ $u(t) \in -\cos\left(\frac{t+33}{5}\right) + [-0.02, 0.02]$

Initial tube $[u](\cdot)$ is obtained with the analytical expression.

Other tubes $[\theta](\cdot)$, $[\dot{x}](\cdot)$, $[x](\cdot)$, \dots , are initialized to $[-\infty, +\infty]$.

Simulation of Mobile Robots

Dead-reckoning

Dead-reckoning: state estimation without observation.

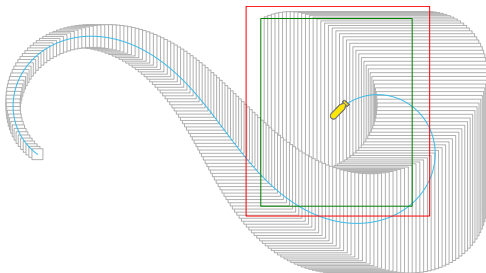


Figure: Interval simulation of the robot following car equations. A strong drift can be observed, depicting cumulative uncertainties.

Simulation of Mobile Robots

Observations

Same simulation considering constraints on both initial and final states:

$$\begin{aligned}\mathbf{x}(0) &\in [-1, 1] \times [-1, 1] \times \left[\frac{-6\pi}{5} - 0.002, \frac{-6\pi}{5} + 0.002\right] \\ \mathbf{x}(14) &\in [53.9, 55.9] \times [6.9, 8.9] \times [-2.36, -2.32]\end{aligned}$$

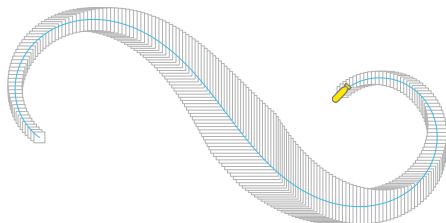


Figure: With a forward/backward contraction, uncertainties are maximal in the middle of the mission.

Simulation of Mobile Robots

Observations

Same simulation adding measurements from beacons.

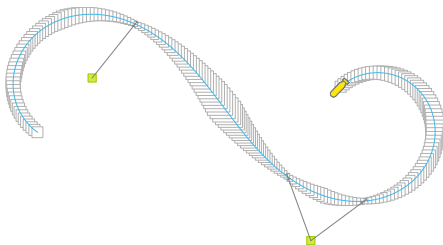


Figure: With a forward/backward contraction, uncertainties are maximal in the middle of the mission.

Simulation of Mobile Robots

Observations

Same simulation without any knowledge on initial or final states:

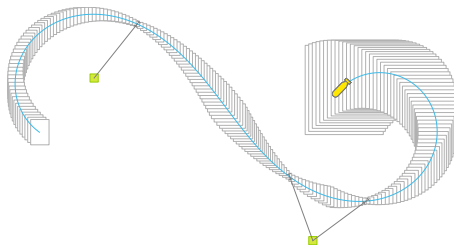


Figure: With a forward/backward contraction, uncertainties are maximal between precise constraints.

Simulation of Mobile Robots

Lissajous example: complex differential equations

Example: a robot described by its state $\mathbf{x}^\top = \{x, y\}$ and following a Lissajous trajectory:

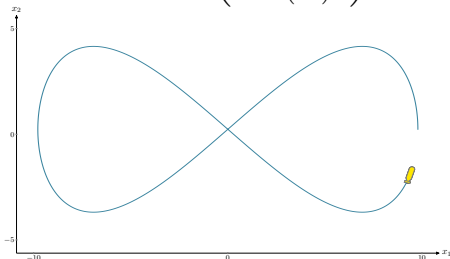
$$\mathbf{x}(t) = 5 \cdot \begin{pmatrix} 2 \cos(t) \\ \sin(2t) \end{pmatrix}$$

Simulation of Mobile Robots

Lissajous example: complex differential equations

Example: a robot described by its state $\mathbf{x}^T = \{x, y\}$ and following a Lissajous trajectory:

$$\mathbf{x}(t) = 5 \cdot \begin{pmatrix} 2 \cos(t) \\ \sin(2t) \end{pmatrix}$$



Initial conditions: $\mathbf{x}(0) \in \begin{bmatrix} 9.8, 10.2 \\ -0.2, 0.2 \end{bmatrix}$, $\dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

Simulation of Mobile Robots

Lissajous example: defining constraints

Defining a set of constraints based on the trajectory equation:

$$\mathbf{x}(t) = \begin{pmatrix} 10 \cos(t) \\ 5 \sin(2t) \end{pmatrix}$$

Simulation of Mobile Robots

Lissajous example: defining constraints

Defining a set of constraints based on the trajectory equation:

$$\mathbf{x}(t) = \begin{pmatrix} 10 \cos(t) \\ 5 \sin(2t) \end{pmatrix}$$

1. Calculating derivatives:

- ▶ $\dot{\mathbf{x}}(t) = \begin{pmatrix} -10 \sin(t) \\ 10 \cos(2t) \end{pmatrix}$
- ▶ $\ddot{\mathbf{x}}(t) = \begin{pmatrix} -10 \cos(t) \\ -20 \sin(2t) \end{pmatrix}$

Simulation of Mobile Robots

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2. Inter-variable relations:

- ▶ $\ddot{x}_2(t) = -20 \sin(t)$

Simulation of Mobile Robots

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2. Inter-variable relations:

- ▶ $\ddot{x}_2(t) = -20 \sin(t)$
- ▶ $\ddot{x}_2(t) = -20 \cdot 2 \cdot \sin(t) \cdot \cos(t)$
- ▶ $\ddot{x}_2(t) = -40 \cdot \frac{\dot{x}_1(t)}{-10} \cdot \frac{\dot{x}_1(t)}{-10}$
- ▶ $\ddot{x}_2(t) = -0.4 \cdot \dot{x}_1(t) \cdot \ddot{x}_1(t)$

Simulation of Mobile Robots

Lissajous example: defining constraints

Defining a set of constraints based on the trajectory equation:

$$\mathbf{x}(t) = \begin{pmatrix} 10 \cos(t) \\ 5 \sin(2t) \end{pmatrix}$$

1. Calculating derivatives:

$$\begin{aligned} \text{▶ } \dot{\mathbf{x}}(t) &= \begin{pmatrix} -10 \sin(t) \\ 10 \cos(2t) \end{pmatrix} \\ \text{▶ } \ddot{\mathbf{x}}(t) &= \begin{pmatrix} -10 \cos(t) \\ -20 \sin(2t) \end{pmatrix} \end{aligned}$$

Selected constraint:

$$\ddot{\mathbf{x}}(t) = \begin{pmatrix} -10 \cos(t) \\ -0.4 \cdot \dot{x}_1 \cdot \ddot{x}_1 \end{pmatrix}$$

2. Inter-variable relations:

$$\begin{aligned} \text{▶ } \ddot{x}_2(t) &= -20 \sin(t) \\ \text{▶ } \ddot{x}_2(t) &= -20 \cdot 2 \cdot \sin(t) \cdot \cos(t) \\ \text{▶ } \ddot{x}_2(t) &= -40 \cdot \frac{\dot{x}_1(t)}{-10} \cdot \frac{\ddot{x}_1(t)}{-10} \\ \text{▶ } \ddot{x}_2(t) &= -0.4 \cdot \dot{x}_1(t) \cdot \ddot{x}_1(t) \end{aligned}$$

Simulation of Mobile Robots

Lissajous example: CSP

The following CSP will be used for state estimation:

$$\left\{ \begin{array}{l} \text{Variables: } \mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot), \ddot{\mathbf{x}}(\cdot) \\ \text{Constraints:} \\ \quad - \ddot{x}_1(t) = -10 \cos(t) \\ \quad - \ddot{x}_2(t) = -0.4 \cdot \dot{x}_1(t) \cdot \ddot{x}_1(t) \\ \quad - \mathbf{x}(0) \in \begin{bmatrix} 9.8, 10.2 \\ -0.2, 0.2 \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ \text{Domains: } [\mathbf{x}](\cdot), [\dot{\mathbf{x}}](\cdot), [\ddot{\mathbf{x}}](\cdot) \end{array} \right.$$

Domains (tubes) are initialized to $[-\infty, \infty] \forall t$.

Simulation of Mobile Robots

Lissajous example: CSP

The following CSP will be used for state estimation:

$$\left\{ \begin{array}{l} \textbf{Variables: } \mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot), \ddot{\mathbf{x}}(\cdot) \\ \textbf{Constraints:} \\ \quad - \ddot{x}_1(t) \in -10 \cos(t) + [-0.001, 0.001] \\ \quad - \ddot{x}_2(t) = -0.4 \cdot \dot{x}_1(t) \cdot \ddot{x}_1(t) \\ \quad - \mathbf{x}(0) \in \begin{bmatrix} 9.8, 10.2 \\ -0.2, 0.2 \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ \textbf{Domains: } [\mathbf{x}](\cdot), [\dot{\mathbf{x}}](\cdot), [\ddot{\mathbf{x}}](\cdot) \end{array} \right.$$

Domains (tubes) are initialized to $[-\infty, \infty] \forall t$.

Simulation of Mobile Robots

Lissajous example: computations

$$\left\{ \begin{array}{l} \text{Variables: } \mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot), \ddot{\mathbf{x}}(\cdot) \\ \text{Constraints:} \\ \quad - \ddot{x}_1(t) \in -10 \cos(t) + [-0.001, 0.001] \\ \quad - \ddot{x}_2(t) = -0.4 \cdot \dot{x}_1(t) \cdot \ddot{x}_1(t) \\ \\ \quad - \mathbf{x}(0) \in \begin{bmatrix} 9.8, 10.2 \\ -0.2, 0.2 \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ \text{Domains: } [\mathbf{x}](\cdot), [\dot{\mathbf{x}}](\cdot), [\ddot{\mathbf{x}}](\cdot) \end{array} \right.$$

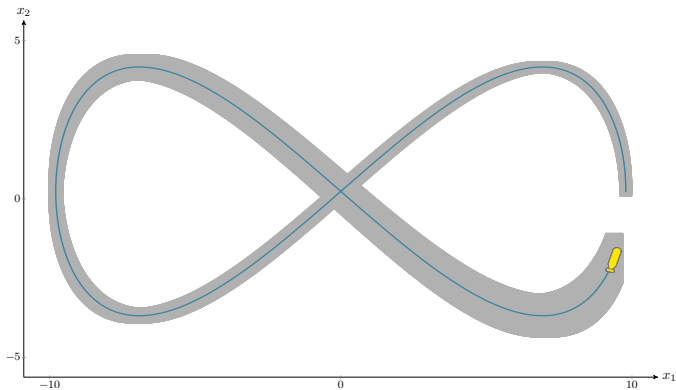


Figure: Set-membership state estimation of a Lissajous robot

Simulation of Mobile Robots

Lissajous example: computations

Other kind of constraints are easily added,
such as periodic constraints: $x_2(t) = x_2(t + \pi)$

$$\left\{ \begin{array}{l} \text{Variables: } \mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot), \ddot{\mathbf{x}}(\cdot) \\ \text{Constraints:} \\ \quad - \ddot{x}_1(t) \in -10 \cos(t) + [-0.001, 0.001] \\ \quad - \ddot{x}_2(t) = -0.4 \cdot \dot{x}_1(t) \cdot \dot{x}_1(t) \\ \quad - x_2(t) = x_2(t + \pi) \\ \quad - \mathbf{x}(0) \in \begin{bmatrix} 9.8, 10.2 \\ -0.2, 0.2 \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \\ \text{Domains: } [\mathbf{x}](\cdot), [\dot{\mathbf{x}}](\cdot), [\ddot{\mathbf{x}}](\cdot) \end{array} \right.$$

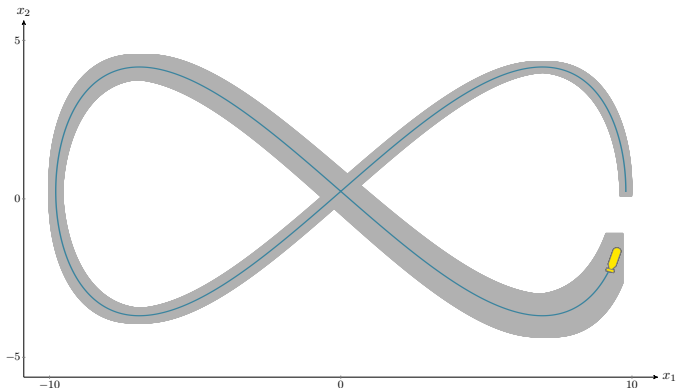


Figure: Set-membership state estimation of a Lissajous robot

Simulation of Mobile Robots

Lissajous example: computations

Other kind of constraints are easily added,
such as periodic constraints: $x_2(t) = x_2(t + \pi)$

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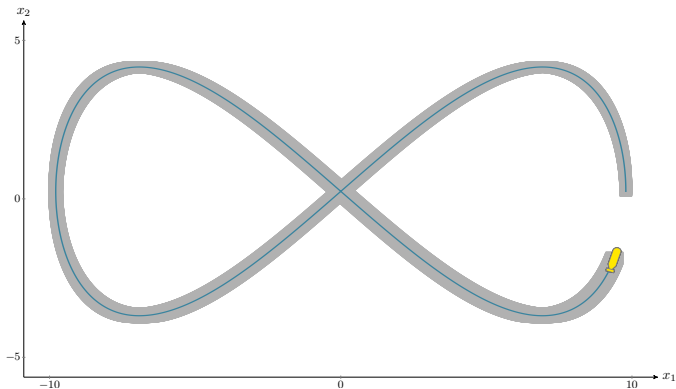


Figure: Set-membership state estimation of a Lissajous robot

Section 5

Application on Real Data-Sets

Application on Real Data-Sets

State estimation of a real underwater robot



Figure: DAURADE AUV managed by DGA Techniques Navales Brest and the Service Hydrographique et Océanographique de la Marine, during an experiment in the Rade de Brest, October 2015.

Application on Real Data-Sets

State estimation of a real underwater robot



Figure: DAURADE AUV managed by DGA Techniques Navales Brest and the Service Hydrographique et Océanographique de la Marine, during an experiment in the Rade de Brest, October 2015.

Application on Real Data-Sets

State estimation of a real underwater robot

Classical kinematic model of an underwater robot:

$$\begin{cases} \dot{\mathbf{p}} &= \mathbf{R}(\psi, \theta, \varphi) \cdot \mathbf{v}_r \\ \dot{\mathbf{v}}_r &= \mathbf{a}_r - \boldsymbol{\omega}_r \wedge \mathbf{v}_r \end{cases}$$

Where:

- ▶ \mathcal{R}_0 is the absolute inertial coordinate system
- ▶ \mathcal{R}_1 is robot's own coordinate system
- ▶ (ψ, θ, φ) and $\mathbf{R}(\psi, \theta, \varphi)$ are Euler angles and matrix
- ▶ $\mathbf{p} = (p_x, p_y, p_z)$ gives the center of the robot in \mathcal{R}_0
- ▶ \mathbf{v}_r is the speed vector in \mathcal{R}_1
- ▶ \mathbf{a}_r is the acceleration vector in \mathcal{R}_1
- ▶ $\boldsymbol{\omega}_r = (\omega_x, \omega_y, \omega_z)$ is the rotation vector of the robot relative to \mathcal{R}_0 expressed in \mathcal{R}_1

Application on Real Data-Sets

State estimation of a real underwater robot

Classical kinematic model of an underwater robot:

$$\begin{cases} \dot{\mathbf{p}} &= \mathbf{R}(\psi, \theta, \varphi) \cdot \mathbf{v}_r \\ \dot{\mathbf{v}}_r &= \mathbf{a}_r - \boldsymbol{\omega}_r \wedge \mathbf{v}_r \end{cases}$$

Sensed values:

Speed vector \mathbf{v}_r , acceleration vector \mathbf{a}_r , Euler angles (ψ, θ, φ)

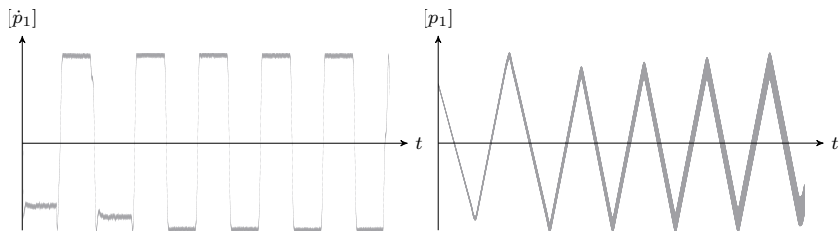
Corresponding tubes:

- ▶ fedded with sensor data:
 $[\mathbf{v}_r](\cdot), [\mathbf{a}_r](\cdot), [\psi](\cdot), [\theta](\cdot), [\varphi](\cdot)$
- ▶ to be computed:
 $[\mathbf{p}](\cdot)$

Application on Real Data-Sets

State estimation of a real underwater robot

Daurade's tubes:



- ▶ $[\dot{p}_1](\cdot)$ – velocity along x in \mathcal{R}_0
tube's thickness coming from measurements remains constant
- ▶ $[p_1](\cdot)$ – position along x in \mathcal{R}_0
tube becomes thicker depicting cumulative errors due to the dead-reckoning method and $[\dot{p}_1](\cdot)$ non-zero thickness

Application on Real Data-Sets

1. Dead-reckoning

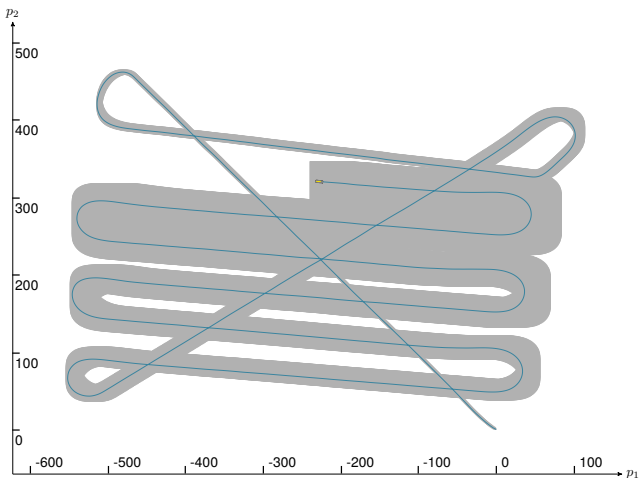


Figure: Mission map: DAURADE exploring a 25 hectares seabed area. STICC

Application on Real Data-Sets

2. Observations (USBL measurements)

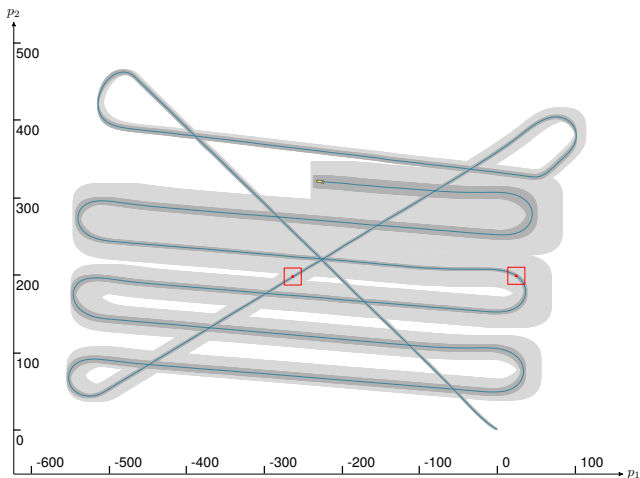


Figure: Mission map: DAURADE exploring a 25 hectares seabed area. STICC

Application on Real Data-Sets

2. Observations (USBL measurements)

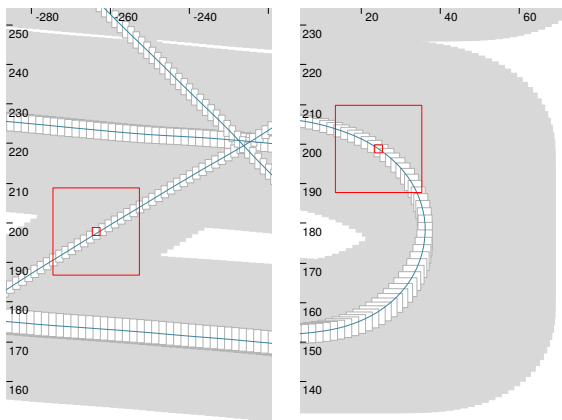


Figure: Zoom on the two state observations: these are bounded positioning measurements obtained at $t_1 = 14$ min and $t_2 = 32$ min.

Conclusion

Tubes and related **contractors** provide a simple and efficient constraint-based method for the computation of **uncertain**, **non-linear** and **dynamical** systems.

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Advantages:

- ▶ simple problem description by constraints definition
- ▶ competitive method for robotics applications
- ▶ easily applicable on data-sets
- ▶ reliable results

Conclusion

Tubes and related **contractors** provide a simple and efficient constraint-based method for the computation of **uncertain**, **non-linear** and **dynamical** systems.

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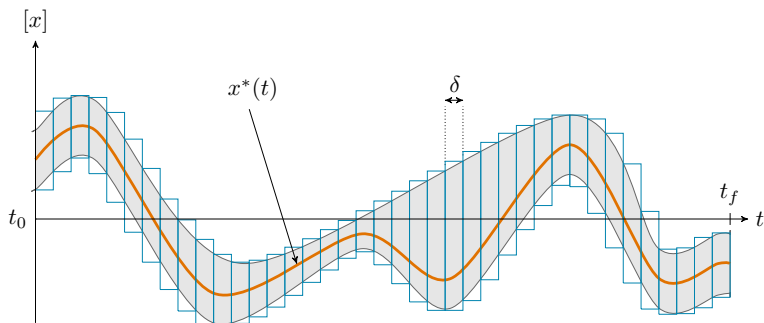
Drawbacks:

- ▶ envelope of solutions only
- ▶ pessimism added by implementation
- ▶ computation time?

Conclusion

An open-source C++/python tube library will be soon available.

`simon.rohou@ensta-bretagne.org`



Support:



DGA

Direction Générale de l'Armement

Tools:



IBEX library

used for interval arithmetic, contractor programming



VIBES

used for rendering

References:

- [Mackworth1977] A. K. MACKWORTH,
Consistency in networks of relations,
Artificial Intelligence, 8(1):99–118, 1977.
- [Chabert2009] G. CHABERT, L. JAULIN,
Contractor Programming,
Artificial Intelligence, 173:1079–1100, 2009.
- [LeBars2012] F. LE BARS, J. SLIWKA, O. REYNET, L. JAULIN,
State estimation with fleeting data,
Automatica, 48(2):381–387, 2012.
- [Bethencourt2014] A. BETHENCOURT, L. JAULIN,
Solving non-linear constraint satisfaction problems involving
time-dependant functions,
Mathematics in Computer Science, 8(3), 2014.
- [Aubry2013] C. AUBRY, R. DESMARE, L. JAULIN,
Loop detection of mobile robots using interval analysis,
Automatica, 2013.
- [Rohou2017] S. ROHOU, L. JAULIN, L. MIHAYLOVA, F. LE BARS, S. M. VERES,
Guaranteed Computation of Robot Trajectories,
Robotics and Autonomous Systems, to be published.

Section 8

Appendix

Appendix

Tubes: implementation

The computer representation of a tube encloses $[x^-(\cdot), x^+(\cdot)]$ inside an interval of step functions $[\underline{x}^-(\cdot), \overline{x}^+(\cdot)]$ such that:

$$\forall t \in \mathbb{R}, \underline{x}^-(t) \leq x^-(t) \leq x^+(t) \leq \overline{x}^+(t)$$

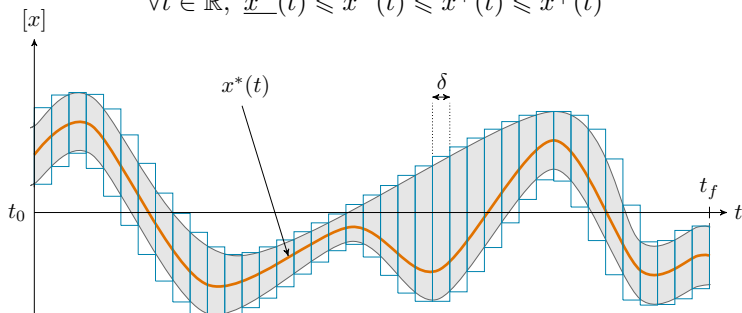


Figure: tube implementation with a set of boxes – this outer representation adds pessimism but enables guaranteed and simple computations

Appendix

Integral of tubes

Definition: the integral of a tube $[x](\cdot) = [x^-, x^+]$ is an interval:

$$\int_a^b [x](\tau) d\tau = \left\{ \int_a^b x(\tau) d\tau \mid x \in [x] \right\} = \left[\int_a^b x^-(\tau) d\tau, \int_a^b x^+(\tau) d\tau \right]$$

[Aubry2013]

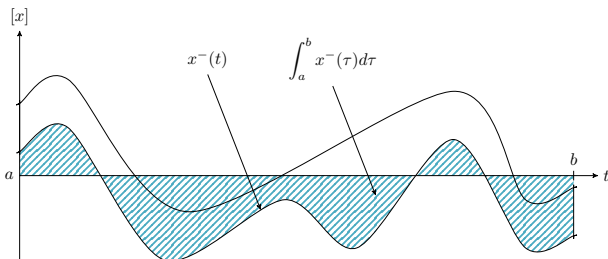


Figure: blue area: lower bound of the tube's integral

Appendix

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Definition: the integral of a tube $[x](\cdot) = [x^-, x^+]$ is an interval:

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[Aubry2013]

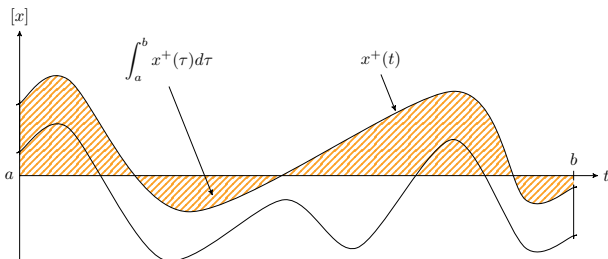


Figure: orange area: upper bound of the tube's integral

Appendix

Definition

A contractor applied on a tube $[a](\cdot)$ aims at removing unfeasible trajectories according to a given constraint \mathcal{L} :

[Bethencourt2014]

$$[a](\cdot) \xrightarrow{\mathcal{C}_{\mathcal{L}}} [b](\cdot)$$

Appendix

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A contractor applied on a tube $[a](\cdot)$ aims at removing unfeasible trajectories according to a given constraint \mathcal{L} :

[Bethencourt2014]

$$[a](\cdot) \xrightarrow{\mathcal{C}_{\mathcal{L}}} [b](\cdot)$$

The output of the contractor $\mathcal{C}_{\mathcal{L}}$ is the tube $[b](\cdot)$ such that:

$$\forall t, [b](t) \subseteq [a](t) \quad (\text{contraction})$$

$$\left(\begin{array}{l} \mathcal{L}(a(\cdot)) \\ a(\cdot) \in [a](\cdot) \end{array} \right) \implies a(\cdot) \in [b](\cdot) \quad (\text{completeness})$$

Appendix

Storyboard of a forward/backward tube contraction

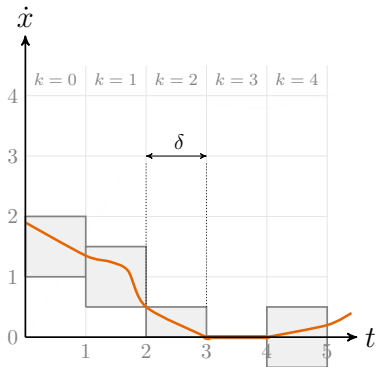


Figure: Derivative tube $[\dot{x}](\cdot)$ used to contract $[x](\cdot)$ initialized to $[-\infty, \infty] \forall t$

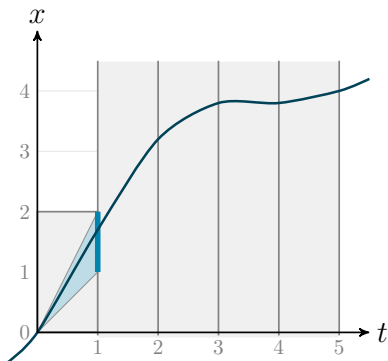


Figure: First step of forward integration contractor: first slice k_0 is contracted from $[-\infty, \infty]$ to $[0, 2]$

Appendix

Storyboard of a forward/backward tube contraction

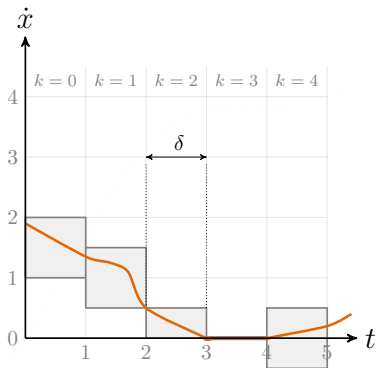


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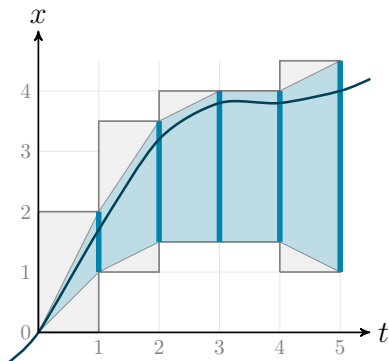


Figure: End of forward contractions. The minimal envelope of solutions compliant with $[\dot{x}](\cdot)$ is pictured in blue. Slices representation – outer approximation – is depicted in gray

Appendix

Storyboard of a forward/backward tube contraction

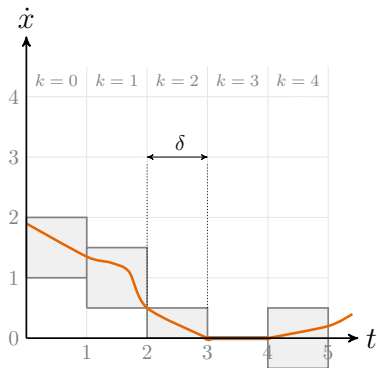


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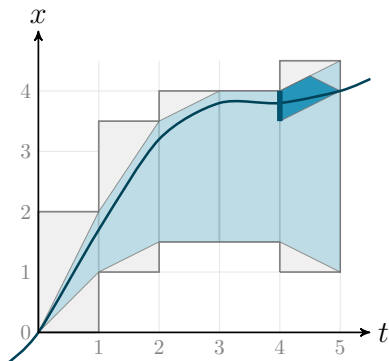


Figure: First step of backward integration contractor, with the last condition $x(5) = 4$. A reduced envelope, pictured in dark blue, is computed from $t = 5$ to $t = 0$

Appendix

Storyboard of a forward/backward tube contraction

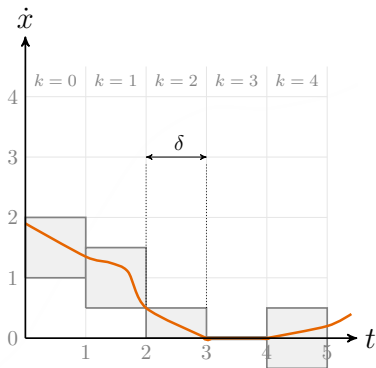


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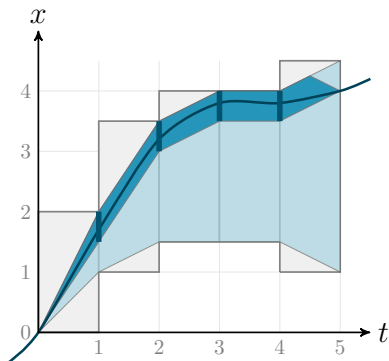


Figure: End of backward processing

Appendix

Storyboard of a forward/backward tube contraction

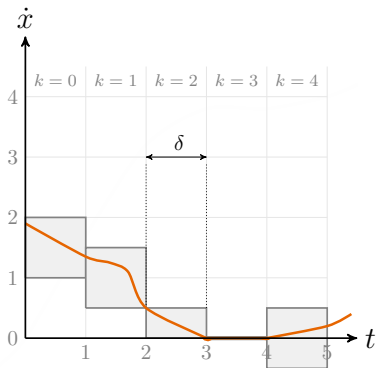


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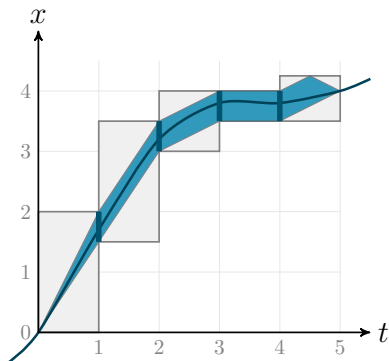


Figure: Tube $[x](\cdot)$ is contracted to optimally enclose the thin envelope pictured in dark blue