An aymptotic minimal contractor for non-linear equations using the Codac library

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Introduction

We consider the problem of approximating the solutions of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^p$ is a non-linear function. In particular, we will consider systems where $p < n$ for which the solution set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$ has infinitely many solutions.

Interval methods can be used to over-approximate such sets in a reliable way. They are often based on axis-aligned boxes $[\mathbf{x}] \in \mathbb{IR}^n$. When sets are described as non-linear systems, such as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, natural inclusion functions can be used to easily and reliably evaluate boxes $[\mathbf{f}]([\mathbf{x}])$. They can be employed in branch-and-prune algorithms in order to pave the solution set $\mathbb{X}$ more accurately. However, these methods involve bisections which comes with an exponential complexity with respect to n. *Contractor* operators, often designed with polynomial complexity, have been shown to overcome this issue by using bisections as a last resort in exploration algorithms [2].

Contractors

A contractor on a set $\mathbb{X}$, denoted by $\mathcal{C}_{\mathbb{X}}$, is an operator that aims at narrowing a box $[\mathbf{x}] \in \mathbb{IR}^n$ in order to reliably remove vectors of $[\mathbf{x}]$

that are not part of the set $\mathbb{X}$. Algorithms exist to automatically build contractors for a given non-linear equation; the state-of-the-art on this topic is the `HC4Revise` algorithm [1].

As for natural inclusion functions, the efficiency of an `HC4Revise` contractor will be impacted by multi-occurrences in the analytic expression of $\mathbf{f}$. Solutions exist, such as symbolic rewriting or affine arithmetic [6], but they are not always minimal, difficult to use, or based on complicated algorithms. On the other hand, the use of centered-form computations provides asymptotically minimal results, as shown in [3].

Centered form contractor

The centered form approach improves the contractions by involving the Jacobian of $\mathbf{f}$ as expressed in Equation (1), where $\overline{\mathbf{x}}$ is the center of the box $[\mathbf{x}]$:

$$\mathbf{f}_c([\mathbf{x}]) = \mathbf{f}(\overline{\mathbf{x}}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big([\mathbf{x}]\big)\big([\mathbf{x}] - \overline{\mathbf{x}}\big). \tag{1}$$

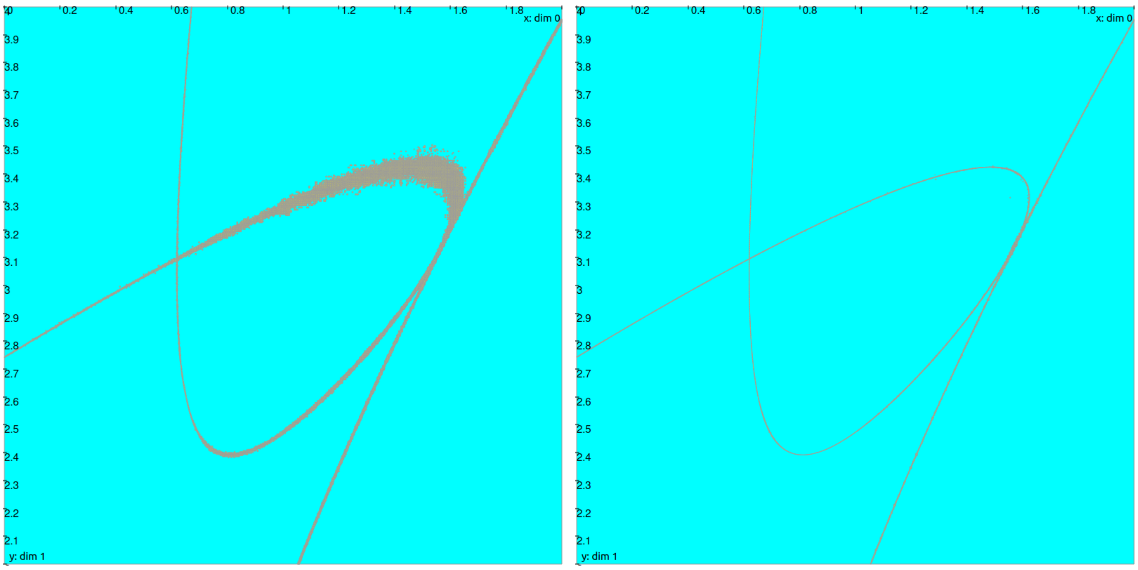
We propose an automatic way to obtain centered form contractors for non-linear systems. First, the interval Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big([\mathbf{x}]\big)$ is computed using Automatic Differentiation. Then, the matricial expression of Equation (1) is treated using an efficient linear contractor with preconditioning.

Implementation and results

The efficiency of the proposed contractor, called `CtcInverse`, will be illustrated on several examples. One of them is taken from the literature [4] and given by the following function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$:

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_3^2 + 2x_3 \sin(x_3 x_1) + \cos(x_3 x_2) \\ 2x_3 \cos(x_3 x_1) - \sin(x_3 x_2) \end{pmatrix}. \tag{2}$$

The solution set $\mathbb{X}$ of Equation (2) for $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is illustrated in Figure 1 for $[\mathbf{x}_0] = [0, 2] \times [2, 4] \times [0, 10]$ and $\epsilon = 4 \times 10^{-3}$. A comparison between the two contractors `HC4Revise` and `CtcInverse` is given: these two algorithms are provided in Codac and are based on the same elementary reverse operations provided by the GAOL library. The computation time difference is mainly due to the number of boxes: `CtcInverse` allows asymptotically minimal contractions for small boxes thanks to the centered form, and so a thinner and thus faster approximation of $\mathbb{X}$.



(a) $\mathbb{X}$ computed with `HC4Revise`. Computation time: 4.51s. 27430 boxes.

(b) $\mathbb{X}$ computed with `CtcInverse`. Computation time: 0.69s. 3713 boxes.

Figure 1: Example of set inversion of Equation (2) using the state-of-the-art `HC4Revise` and the proposed `CtcInverse` contractors. The approximated three-dimensional solution sets are projected onto (x_1, x_2).

`CtcInverse` is now available in the Codac library [5] (v2.0). In particular, the code of Figure 1b is given in Figure 2 as an example of use of Codac. The user does not have to provide the Jacobian of Equation (1), it is deduced by Automatic Differentiation. Codac is available in C++, Python and Matlab languages and provided under GNU LGPL. More information on: `http://codac.io`

```
from codac import * # using Codac 2.0 at least

x = VectorVar(3)
f = AnalyticFunction([x], vec(
  -sqr(x[2])+2*x[2]*sin(x[2]*x[0])+cos(x[2]*x[1]),
  2*x[2]*cos(x[2]*x[0])-sin(x[2]*x[1])
))

ctc = CtcInverse(f, [[0],[0]])
pave([[0,2],[2,4],[0,10]], ctc, 0.004)
```

Figure 2: Inversion of Eq. (2) using the Codac library (here in Python).

References

[1] F. Benhamou, F. Goualard, L. Granvilliers, and J. F. Puget. Revising hull and box consistency. In *Proceedings of the International Conference on Logic Programming*, pages 230–244, 1999.

[2] Gilles Chabert and Luc Jaulin. Contractor programming. *Artificial Intelligence*, 173(11):1079–1100, July 2009.

[3] Luc Jaulin. Asymptotically minimal interval contractors based on the centered form; application to the stability analysis of linear time-delayed differential equations. *Acta Cybernetica*, 2024.

[4] Rachid Malti, Milan R. Rapaić, and Vukan Turkulov. A unified framework for exponential stability analysis of irrational transfer functions in the parametric space. *Annual Reviews in Control*, 57:100935, 2024.

[5] Simon Rohou, Benoît Desrochers, and Fabrice Le Bars. The Codac Library. *Acta Cybernetica*, Mar. 2024.

[6] Siegfried M. Rump and Masahide Kashiwagi. Implementation and improvements of affine arithmetic. *Nonlinear Theory and Its Applications, IEICE*, 6(3):341–359, 2015.